Lecture 20

10 Nov 2015

Last time: integrals and calculating them using FTC.

FTC:

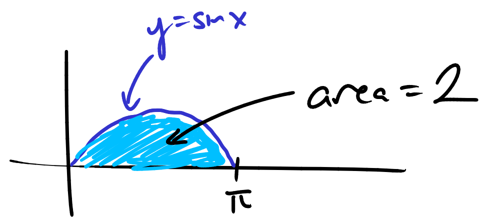
I. $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

i.e. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

II. $\int_a^b f(x)\,dx = F(b) - F(a) = F\Big|_a^b$ where $F(x)$ is any antiderivative of $f(x)$.

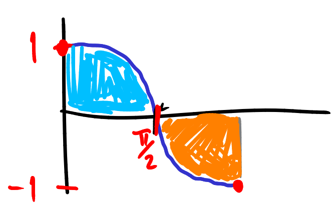
Ex $\frac{d}{dx}\int_3^x \cos(t^7)\,dt = \cos(x^7)$. (FTC I)

Ex $\int_0^\pi \sin x\,dx = 2$ (last time)



Ex $\int_0^\pi \cos x\,dx = ?$ and what does it mean in terms of areas?

$\int_0^\pi \cos x\,dx = \sin x\Big|_0^\pi = \sin(\pi) - \sin(0) = 0 - 0 = \underline{\underline{0}}$.



[i.e. the area of blue, orange regions are equal, both $= A$ $(=1)$
total integral is $A - A = \underline{\underline{0}}$]

[**Rk** could also look at $\int_0^\pi |\cos x|\,dx$ to get the actual area

— to do this, break it up piecewise $|\cos x| = \begin{cases} \cos x & 0 \le x \le \pi/2 \\ -\cos x & \pi/2 \le x \le \pi \end{cases}$

so $\int_0^\pi |\cos x|\,dx = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx = \cdots = 1 + 1 = 2$.]

Ex Calculate $\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta \, d\theta$.

Use FTC II. $\sec\theta$ is an antiderivative of $\sec\theta\tan\theta$. (because $\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$)

So:

$$\int_{\pi/4}^{\pi/3} \sec\theta\tan\theta \, d\theta = \sec\theta \Big|_{\pi/4}^{\pi/3} = \sec\tfrac{\pi}{3} - \sec\tfrac{\pi}{4}$$

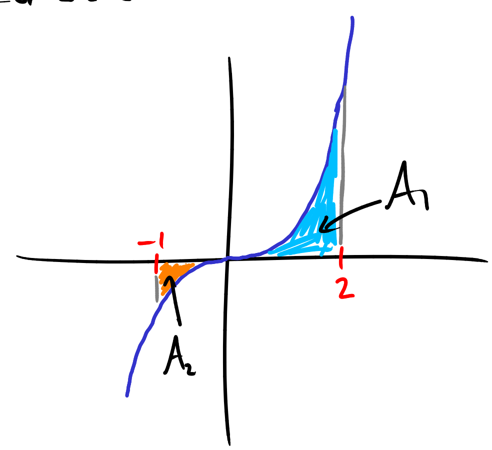
$$= \underline{\underline{2 - \sqrt{2}}}$$

Ex Calculate $\int_{-1}^{2} x^3 \, dx$ and interpret it as a difference of areas

$$\int_{-1}^{2} x^3 \, dx = \frac{1}{4}x^4 \Big|_{-1}^{2} = \frac{1}{4}(2)^4 - \frac{1}{4}(-1)^4$$

$$= 4 - \frac{1}{4} = \frac{15}{4}$$

i.e. $\underline{\underline{A_1 - A_2 = \frac{15}{4}}}$



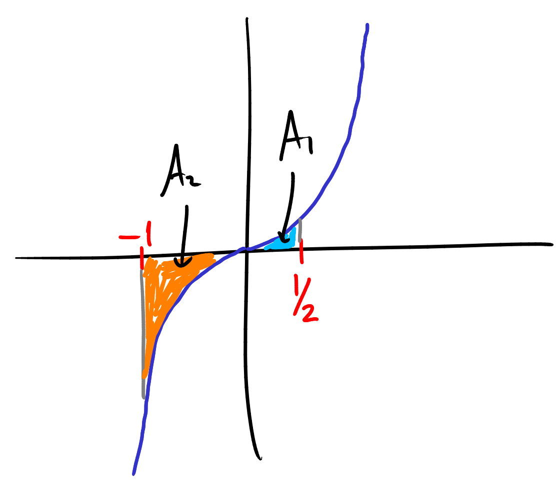
(Remark: if we use another antideriv, still get same answer, e.g.

$$\int_{-1}^{2} x^3 \, dx = \frac{1}{4}x^4 + 1 \Big|_{-1}^{2} = \cdots = 5 - \frac{5}{4} = \underline{\frac{15}{4}} \Big)$$

Ex Is $\int_{-1}^{1/2} \tan x \, dx$ positive, negative or zero?

$$\int_{-1}^{1/2} \tan x \, dx = A_1 - A_2 < 0$$

since A_2 is bigger than A_1.

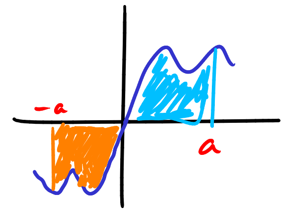


Ex Is $\int_{-1/2}^{1/2} \tan x \, dx$ positive, negative or zero? It's $\underline{\text{zero}}$.

General rule: integrals of symmetric functions

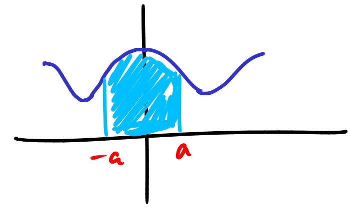
a) If f is odd, $f(-x) = -f(x)$

then $\int_{-a}^{a} f(x)\,dx = 0$



b) If f is even, $f(x) = f(-x)$

then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



Ex
$$\int_{-0.154}^{0.154} \frac{\overset{\text{odd}}{(\tan x)}\,\overset{\text{even}}{\left(x^6+29x^4+\frac{105}{3}x^2+981.23\right)}}{\underbrace{x^{12}+7\cos(32x)}_{\text{even}}}\,dx = 0$$

Ex
$$\int_{\pi/6}^{\pi/3} \left(-\frac{3}{\sin^2\theta}+\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/3} \left(-3\csc^2\theta+\theta\right) d\theta$$

$$= 3\cot\theta + \frac{\theta^2}{2}\Bigg|_{\pi/6}^{\pi/3}$$

$$= \cdots = \underline{\underline{-2\sqrt{3}+\frac{\pi^2}{24}}}$$

Indefinite integrals

Notation: $\int f(x)\,dx$ means any antiderivative of $f(x)$.

Ex $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$\int \sin x\,dx = -\cos x + C$ $\qquad$ $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$ $\qquad$ $\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$ $\qquad$ $\int \csc x \cot x\,dx = -\csc x + C$

Ex Find $\int (10x^4 + 6\sec^2 x)\,dx$.

$$= 10\left(\frac{x^5}{5}\right) + 6 \cdot \tan x + C = \underline{\underline{2x^5 + 6\tan x + C}}$$

Ex Find $\int_0^{\pi/4} (10x^4 + 6\sec^2 x)\,dx$

$$= 2x^5 + 6\tan x \Big|_0^{\pi/4}$$

$$= \left[2\left(\frac{\pi}{4}\right)^5 + 6\tan\frac{\pi}{4}\right] - \left[2\cdot(0)^5 + 6\cdot\tan(0)\right]$$

$$= \frac{\pi^5}{512} + 6 \qquad - \qquad 0$$

$$= \underline{\underline{\frac{\pi^5}{512} + 6}}$$

Ex Find $\int u^{2/3}\,du$. $\qquad n = \frac{2}{3} \quad n+1 = \frac{5}{3}$

$$\frac{u^{5/3}}{\frac{5}{3}} + C = \underline{\underline{\frac{3}{5}u^{5/3} + C}}$$

Ex Find $\int_1^8 u^{2/3}\,du$

$$= \frac{3}{5}u^{5/3}\Big|_1^8 = \frac{3}{5}(8^{5/3}) - \frac{3}{5}(1^{5/3}) = \frac{3}{5}(2^5) - \frac{3}{5}(1) = \frac{3}{5}(32-1)$$

$$= \underline{\underline{\frac{93}{5}}}$$

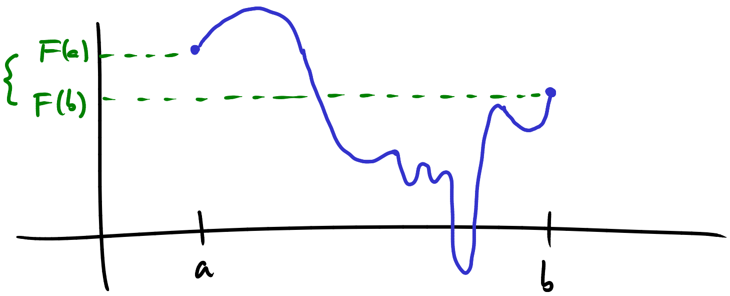
Net change

Given a function $F(t)$ t = time

$F'(t)$ is the rate of change of $F(t)$.

$$\int_a^b F'(t)\,dt = F(b) - F(a) = \text{net change of } F \text{ over time interval } [a,b].$$

$F(b) - F(a) = \{$ net change



Ex Water flows into a reservoir at the rate $(10t+6)\ ft^3/s$.   (t in sec)

The reservoir contains $400\ ft^3$ of water at time $t=0$.

How much does it contain at time $t = 10\,s$?

The net change from $t=0$ to $t=10$ is

$$\int_0^{10} (10t+6)\,dt = 5t^2 + 6t \Big|_0^{10}$$

$$= \left(5(10^2) + 6(10)\right) - \left(5(0^2) + 6(0)\right)$$

$$= (500 + 60) \quad - \quad (0 + 0)$$

$$= 560\ ft^3$$

So the amount at time $t=10$ is $400 + 560 = \underline{\underline{960\ ft^3}}$.

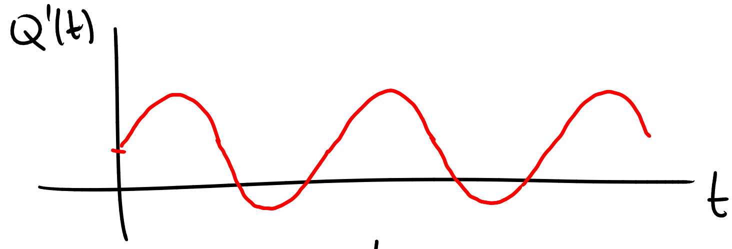
Ex A capacitor   is connected to a load that can charge or discharge it.

The current flowing into the cap. is $Q'(t) = \sin(\pi t) + \frac{1}{2}$.

($Q(t)$ = charge of battery at time t)

If the cap. starts with 10 units of charge at $t=0$ $(Q(0) = 10)$

how much does it have at $t=6$?



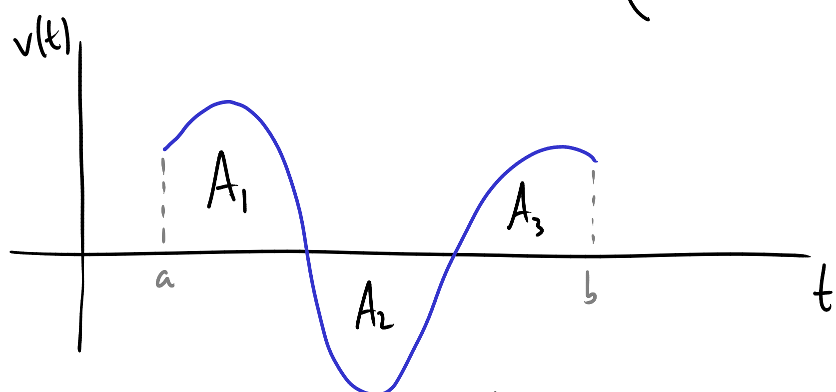
$$Q(6) - Q(0) = \int_0^6 Q'(t)\,dt$$
$$= \int_0^6 \left(\sin(\pi t) + \tfrac{1}{2}\right) dt$$
$$= -\frac{1}{\pi}\cos(\pi t) + \frac{t}{2} \Big|_0^6$$
$$= \left(-\tfrac{1}{\pi}\cos(6\pi) + 3\right) - \left(-\tfrac{1}{\pi}\cos(0\pi) + 0\right)$$
$$= \left(-\tfrac{1}{\pi} + 3\right) - \left(-\tfrac{1}{\pi} + 0\right)$$
$$= 3$$

so $Q(6) = 3 + Q(0) = 3 + 10 = \underline{\underline{13}}$.

A standard example of net change: <u>total displacement</u>.

If $s(t)$ = position along a line
$s'(t) = v(t)$ velocity

$\left(\begin{array}{ll} v(t)>0: & \text{moving to the right} \\ v(t)<0: & \text{" " " left} \end{array} \right)$



<u>Total displacement</u> $s(b) - s(a) = \int_a^b v(t)\,dt = A_1 - A_2 + A_3$

<u>Total distance</u> (odometer reading) $\int_a^b |v(t)|\,dt = A_1 + A_2 + A_3$

Ex A particle moves along a line with $v(t) = t^2 - t - 6$ m/s, from time $t=1$ to $t=4$.

a) What is the total displacement of the particle?

$$\Delta s = s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 t^2 - t - 6\,dt$$

$$= \left.\frac{t^3}{3} - \frac{t^2}{2} - 6t\,\right|_1^4$$

$$= \cdots = -\frac{9}{2}$$

(i.e. $\frac{9}{2}$ m to the left (negative direction))

b) What is the total distance it travels?

$$\int_1^4 |v(t)|\,dt \qquad v(t) = (t-3)(t+2)$$

$$\int_1^4 |v(t)|\,dt = \int_1^3 |v(t)|\,dt + \int_3^4 |v(t)|\,dt$$

$$= \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt \qquad = \cdots$$