Lecture 21

12 Nov 2015

Last time: def, indef $\int$ and Net Change Theorem

Ex A particle moves along a line with $v(t) = t^2 - t - 6$ m/s (t in s) from time $t=1$ to $t=4$.

a) What is the total displacement?

$$\Delta s = s(4) - s(1) = \int_1^4 v(t)\,dt$$

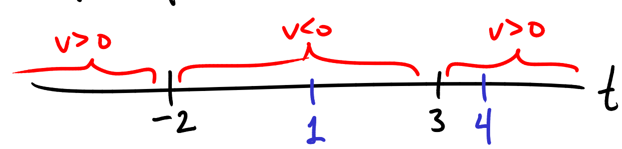
$$= \int_1^4 (t^2 - t - 6)\,dt = \cdots = -\frac{9}{2}$$

(i.e. $\frac{9}{2}$ m to the left/negative dir)

b) What is the total distance the particle covers?

$$\int_1^4 |v(t)|\,dt$$

$$v(t) = (t-3)(t+2)$$



$$\int_1^4 |v(t)|\,dt = \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 -t^2 + t + 6\,dt + \int_3^4 t^2 - t - 6\,dt$$

$$= -\frac{1}{3}t^3 + \frac{1}{2}t^2 + 6t\,\Big|_1^3 + \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\,\Big|_3^4$$

$$= \cdots$$

$$= \frac{22}{3} + \frac{17}{6} = \underline{\underline{\frac{61}{6}}} \quad \text{i.e. } \frac{61}{6}\text{ m}$$

Method of substitution ("u-substitution")

A method of finding antiderivatives.

Ex $\int \sqrt{2x-3}\,dx = ?$

Try to relate this to something easier to understand: introduce $u = 2x-3$

Replace x by u everywhere.

$$\int \sqrt{2x-3}\,dx = \int \sqrt{u}\,dx$$

To relate dx to du: $\frac{du}{dx} = 2$, so $du = 2\,dx$

so $\frac{1}{2}du = dx$

$$\text{so} \quad \int \sqrt{u}\,dx = \int \sqrt{u}\cdot \tfrac{1}{2}\,du$$

$$= \tfrac{1}{2}\int \sqrt{u}\,du$$

$$= \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2} + C$$

$$= \tfrac{1}{3}u^{3/2} + C$$

$$= \underline{\underline{\tfrac{1}{3}(2x-3)^{3/2} + C}}$$

Ex $\int 7x e^{x^2}\,dx = ?$

Set $u = x^2$.

Then $e^{x^2} = e^u$

and $\frac{du}{dx} = 2x$, so $du = 2x\,dx$, i.e. $\frac{1}{2}du = x\,dx$

$$\text{then} \quad \int 7x e^{x^2}\,dx = 7\int e^{x^2}\cdot x\,dx$$

$$= 7\int e^u \cdot \tfrac{1}{2}\,du$$

$$= \tfrac{7}{2}\int e^u\,du$$

$$= \frac{7}{2} e^{u} + C$$

$$= \frac{7}{2} e^{x^2} + C$$

Ex $$\int \frac{x^2+16x+8}{\sqrt{\frac{x}{2}+1}}\, dx = ?$$

Could try $u = \sqrt{\frac{x}{2}+1}$

$$du = \frac{\frac{1}{4}}{\sqrt{\frac{x}{2}+1}}\, dx = \frac{1}{4u}\, dx$$

$$4u\, du = dx$$

$$= \int \frac{x^2+16x+8}{u}\, 4u\, du$$

then to subst. back in for x: $u^2 = \frac{x}{2}+1$

$$2u^2 = x+2$$

$$x = 2u^2-2$$

$$= \int \frac{(2u^2-2)^2+16(2u^2-2)+8}{u}\, 4u\, du$$

$$= 4\int (2u^2-2)^2 + 16(2u^2-2)+8\, du$$

$$= 4\int 4u^4 - 8u^2 + 4 + 32u^2 - 32 + 8\, du$$

$$= 4\int 4u^4 + 24u^2 - 20\, du$$

$$= 4\left(\frac{4}{5}u^5 - 8u^3 - 20u\right) + C$$

and substitute back $u = \sqrt{\frac{x}{2}+1}$ . . .

finally get $$= \frac{4}{5}\sqrt{\frac{x}{2}+1}\,\left(x^2+24x-56\right)$$

Can also do this $\int$ by substituting $u = \frac{x}{2} + 1$.

$$\int \frac{x^2+16x+8}{\sqrt{\frac{x}{2}+1}}\,dx$$

$u = \frac{x}{2}+1 \qquad x = 2u-2$

$du = \frac{1}{2}dx \quad$ i.e $\quad dx = 2du$

$$= \int \frac{(2u-2)^2 + 16(2u-2) + 8}{\sqrt{u}}(2du)$$

$$= \dots$$

Substitution for <u>definite</u> integrals: very similar to indefinite, but have to remember to transform the <u>limits</u> too!

<u>Ex</u> $\displaystyle\int_0^{\pi/2} \sin(2x)\,dx$

$u = 2x \qquad x = \frac{u}{2}$

$dx = \frac{1}{2}du$

$$= \int_{x=0}^{x=\pi/2} \sin(2x)\,dx$$

$$= \int_{u=0}^{u=\pi} \sin(u)\cdot\left(\tfrac{1}{2}du\right)$$

$$= \frac{1}{2}\int_0^{\pi} \sin(u)\,du \qquad = \frac{1}{2}\left(-\cos(u)\Big|_0^{\pi}\right)$$

$$= \frac{1}{2}\left((-\cos\pi) - (-\cos 0)\right)$$

$$= \frac{1}{2}\left(-(-1) - (-1)\right) = \frac{1}{2}(1+1) = \underline{\underline{1}}$$

<u>Ex</u> $\displaystyle\int_{\pi/3}^{\pi/2} (\cos 3x)\, e^{(\sin 3x)}\,dx$

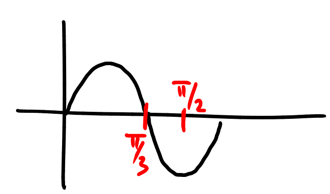
$u = \sin 3x$

$du = 3\cos 3x\,dx \longrightarrow \frac{1}{3}du = (\cos 3x)\,dx$

$$= \int_0^{-1} e^u \cdot \frac{1}{3}du$$

$x = \pi/3 \rightarrow u = \sin 3\left(\frac{\pi}{3}\right) = \sin\pi = 0$

$x = \pi/2 \rightarrow u = \sin 3\left(\frac{\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$



$$= \frac{1}{3}\int_0^{-1} e^u\,du$$

$$= \frac{1}{3}\left(e^u \Big|_0^{-1}\right)$$

$$= \frac{1}{3}\left(e^{-1}-1\right)$$

Remark: instead of transforming the limits, can also put the indefinite $\int$ back in terms of x and then use the original limits. e.g. in this example,

$$= \int_{x=\pi/3}^{x=\pi/2} e^u \cdot \frac{1}{3}\, du$$

$$= \frac{1}{3}\left(e^u \Big|_{x=\pi/3}^{x=\pi/2}\right)$$

$$= \frac{1}{3}\left(e^{\sin 3x} \Big|_{x=\pi/3}^{x=\pi/2}\right)$$

$$= \frac{1}{3}\left(e^{-1}-1\right)$$

Ex $\int \sqrt{1+x^2}\; x^5\, dx$

$u = 1+x^2$
$du = 2x\, dx$
$\frac{1}{2} du = x\, dx$

$$= \int \sqrt{u} \cdot x^4 \cdot x\, dx$$

$$= \int \sqrt{u} \cdot x^4 \cdot \frac{1}{2}\, du$$

and $x^2 = u-1$
so $x^4 = (u-1)^2$

$$= \int \sqrt{u} \cdot (u-1)^2 \cdot \frac{1}{2}\, du$$

$$= \frac{1}{2}\int u^{1/2}\left(u^2-2u-1\right)\, du$$

$$= \frac{1}{2}\int u^{5/2} - 2u^{3/2} - u^{1/2}\, du$$

$$= \dots$$

Ex $\int \tan x \, dx$

$u = \cos x$
$du = -\sin x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

$= \int \frac{\sin x \, dx}{\cos x}$

$= \int \frac{-du}{u}$

$= -\ln|u| + C$

$= -\ln|\cos x| + C$

$= \ln|\cos x|^{-1} + C$

$= \underline{\ln|\sec x| + C}$

Side remark:
$\frac{1}{x}$ has antideriv. $\ln x$ $\underline{\text{if } x>0}$
if we want to allow also $x<0$,
then antideriv. of $\frac{1}{x}$ is $\ln|x|$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}} dx$ $\qquad 2du = \frac{dx}{\sqrt{x}}$

$= \int e^u \cdot 2 du$

$= 2e^u + C \quad = 2e^{\sqrt{x}} + C$

(Check: $\frac{d}{dx}(2e^{\sqrt{x}} + C) = 2 \cdot e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}$)

$$\int \frac{dx}{\sqrt{x}\,(1+x)}$$

try $u = 1+x$:
$du = dx$
$x = u-1$

$$\longrightarrow \int \frac{du}{\sqrt{x}\cdot u} = \int \frac{du}{\sqrt{u-1}\cdot u}$$

<u>not helping</u>

try $u = \sqrt{x}$:

$du = \frac{1}{2\sqrt{x}}\,dx \qquad 2du = \frac{dx}{\sqrt{x}}$

$$\longrightarrow \quad \int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2du}{1+x}$$

and $u^2 = x$, so

$$= \int \frac{2\,du}{1+u^2} = 2\tan^{-1}(u) + C$$

$$= \underline{2\tan^{-1}(\sqrt{x}) + C}$$