Lecture 21

12 Nov 2015

Last time: def, indef $\int$ and Net Change Theorem

Ex A particle moves along a line with velocity $v(t) = t^2 - t + 6$ m/s (time in s) from time $t=1$ to $t=4$.

a) What is its total displacement?

$$\Delta s = s(4) - s(1) = \int_1^4 v(t)\,dt$$

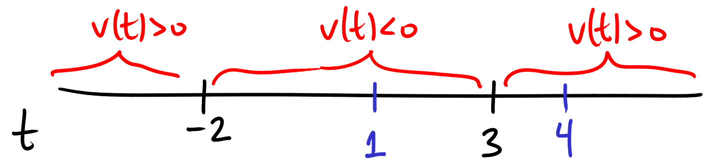
$$= \int_1^4 (t^2 - t + 6)\,dt = \cdots = -\frac{9}{2}$$

i.e. $\frac{9}{2}$m in the negative direction (left)

b) What is the total distance the particle covers?

$$\int_1^4 |v(t)|\,dt$$

$$v(t) = t^2 - t - 6 = (t-3)(t+2)$$



$$\int_1^4 |v(t)|\,dt = \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 -t^2 + t + 6\,dt + \int_3^4 t^2 - t - 6\,dt$$

$$= -\frac{1}{3}t^3 + \frac{1}{2}t^2 + 6t\Big|_1^3 + \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\Big|_3^4$$

$$= \cdots$$

$$= 22/3 + 17/6 = \underline{\underline{\frac{61}{6}}} \quad \text{i.e. } \frac{61}{6}\text{ m}.$$

(check: $-22/3 + 17/6 = \frac{-44+17}{6} = \frac{-27}{6} = -\frac{9}{2}$ ✓)

Method of substitution ("u-substitution")

A method of finding antiderivatives.

Ex $\int \sqrt{2x-3}\, dx = ?$

Introduce new variable $u = 2x-3$

$$\int \sqrt{2x-3}\, dx = \int \sqrt{u}\, dx$$

To relate dx and du: $u = 2x-3$ $\quad \frac{du}{dx} = 2$

$$du = 2\, dx$$
$$\tfrac{1}{2} du = dx$$

$$\int \sqrt{u}\, dx = \int \sqrt{u} \cdot \left(\tfrac{1}{2} du\right)$$
$$= \frac{1}{2} \int \sqrt{u}\, du$$
$$= \frac{1}{2} \cdot \left(\frac{2}{3} u^{3/2}\right) + C$$
$$= \frac{1}{3} u^{3/2} + C$$
$$= \frac{1}{3} (2x-3)^{3/2} + C$$

$\int f(x)\, dx =$ antiderivative of f.

Ex $\int 7xe^{x^2} dx$

try $u = x^2$
$du = 2x\, dx \quad \frac{1}{2} du = x\, dx$

$$= 7 \int e^u x\, dx$$
$$= 7 \int e^u \cdot \tfrac{1}{2} du$$
$$= \frac{7}{2} \int e^u du$$
$$= \frac{7}{2} e^u + C$$
$$= \frac{7}{2} e^{x^2} + C.$$

Check: $\frac{d}{dx}\left(\frac{7}{2} e^{x^2} + C\right) = \frac{7}{2} \cdot \left(2xe^{x^2}\right)$
$= 7xe^{x^2}$ ✓

Ex $\displaystyle\int \frac{x^2+16x+8}{\sqrt{\frac{x}{2}+1}}\,dx = ?$ Two possibilities: $u=\sqrt{\frac{x}{2}+1}$

$u=\frac{x}{2}+1$

Either works. Let's take $u=\frac{x}{2}+1$.

$du=\frac{1}{2}dx \qquad dx=2du$

$$\int \frac{x^2+16x+8}{\sqrt{u}}\,dx$$

$$=\int \frac{x^2+16x+8}{\sqrt{u}}\cdot(2\,du)$$

and use $2u=x+2$

$x=2u-2$

$$=2\int \frac{(2u-2)^2+16(2u-2)+8}{\sqrt{u}}\,du$$

$$=2\int \frac{(4u^2-8u+4)+32u-32+8}{\sqrt{u}}\,du$$

$$=2\int \frac{4u^2+24u-20}{\sqrt{u}}\,du = 2\int 4u^{3/2}+24u^{1/2}-20u^{-1/2}\,du$$

$$=2\left(4\cdot\tfrac{2}{5}u^{5/2}+24\cdot\tfrac{2}{3}u^{3/2}-20\cdot 2u^{1/2}\right)+C$$

$$=\frac{16}{5}u^{5/2}+\frac{96}{3}u^{3/2}-80u^{1/2}+C$$

and substitute back $u=\frac{x}{2}+1$

to get final answer in terms of x:

$$\ldots = \frac{4}{5}\sqrt{\frac{x}{2}+1}\left(x^2+24x-56\right)+C$$

Substitution for definite integrals: very similar to indefinite $\int$, but have to remember to transform the limits of integration too!

Ex $\displaystyle\int_0^{\pi/2} \sin(2x)\,dx$

$u=2x$

$du=2\,dx$

$\frac{1}{2}du=dx$

$$= \int_{x=0}^{x=\pi/2} \sin(2x)\,dx$$

$$= \int_{x=0}^{x=\pi/2} \sin(u)\cdot \tfrac{1}{2}\,du$$

$$= \tfrac{1}{2}\int_{u=0}^{u=\pi} \sin(u)\,du$$

$$= \tfrac{1}{2}\left(-\cos(u)\Big|_{u=0}^{u=\pi}\right)$$

$$= \tfrac{1}{2}\left((-\cos\pi) - (-\cos 0)\right)$$

$$= \tfrac{1}{2}(1+1) = \underline{1}$$

<u>Ex</u> $\displaystyle\int_{x=\pi/3}^{x=\pi/2} (\cos 3x)\, e^{(\sin 3x)}\,dx$

$u = \sin 3x$

$du = 3\cos 3x\,dx \longrightarrow \tfrac{1}{3}du = \cos 3x\,dx$

$$= \int_{u=0}^{u=-1} e^u \cdot \tfrac{1}{3}\,du$$

$x = \pi/3 \longrightarrow u = \sin(3\cdot \pi/3) = \sin(\pi) = 0$

$x = \pi/2 \longrightarrow u = \sin(3\cdot \pi/2) = \sin\left(\frac{3\pi}{2}\right) = -1$

$$= \tfrac{1}{3} e^u \Big|_{u=0}^{u=-1}$$

$$= \tfrac{1}{3}\left(e^{-1} - e^0\right) = \underline{\underline{\tfrac{1}{3}\left(\tfrac{1}{e} - 1\right)}}$$

<u>Ex</u> $\displaystyle\int \sqrt{1+x^2}\cdot x^5\,dx$

$u = 1+x^2$

$du = 2x\,dx \qquad \tfrac{1}{2}du = x\,dx$

$$= \int \sqrt{u}\cdot x^5\,dx$$

$$= \int \sqrt{u}\cdot x^4\cdot x\,dx$$

$$= \int \sqrt{u}\cdot x^4\cdot \tfrac{1}{2}\,du$$

$u = 1+x^2 \longrightarrow x^2 = u-1$

$x^4 = (u-1)^2$

$$= \int \sqrt{u}\cdot (u-1)^2\cdot \tfrac{1}{2}\,du$$

$= \frac{1}{2}\int u^{1/2}(u^2-2u+1)\,du$

$= \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\,du$

$= \ldots$

<u>Ex</u> $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)}$ try $u = 1+x$: $du = dx$

$\displaystyle = \int \frac{du}{\sqrt{x}\cdot u}$ $u = 1+x$, $x = u-1$

$\displaystyle = \int \frac{du}{\sqrt{u-1}\cdot u}$ $\longrightarrow$ <u>didn't help</u>. try $u = \sqrt{x}$ instead.

$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{dx}{\sqrt{x}}$

$\displaystyle \int \frac{dx}{\sqrt{x}(1+x)}$

$\displaystyle = \int \frac{2\,du}{1+x}$ and $u = \sqrt{x}$, so $x = u^2$

$\displaystyle = \int \frac{2\,du}{1+u^2}$

$= 2\tan^{-1}(u) + C$

$= \underline{\underline{2\tan^{-1}(\sqrt{x}) + C}}$

<u>Ex</u> $\displaystyle\int \tan x\,dx$

$\displaystyle = \int \frac{\sin x}{\cos x}\,dx$ $u = \cos x$, $du = -\sin x\,dx$, $-du = \sin x\,dx$

$\displaystyle = \int \frac{-du}{u}$

$$= -\ln|u| + C$$
$$= -\ln|\cos x| + C$$
$$= \ln|\cos x|^{-1} + C$$
$$= \underline{\underline{\ln|\sec x| + C}}$$

$$\int \frac{1}{x}\,dx = \ln|x|$$

<u>Ex</u> $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = ?$

$u = \sqrt{x}$

$du = \dfrac{1}{2\sqrt{x}}\,dx$

$2\,du = \dfrac{dx}{\sqrt{x}}$

$$= \int e^{u} \cdot 2\,du$$

$$= 2e^{u} + C = \underline{\underline{2e^{\sqrt{x}} + C}}$$