Lecture 22

17 Nov 2015

Q from HW:

Given $F(x) = \int_4^x 3t^2 - 4t \, dt$

how to tell when $F(x)$ is concave down?

$F'(x) = 3x^2 - 4x$

$F''(x) = 6x - 4$

$\vdots$

If $F(x) = \int_{-7}^{x^4} \sin(t)\, dt$

$F'(x) = \sin(x^4) \cdot \underbrace{4x^3}_{\text{from chain rule}}$

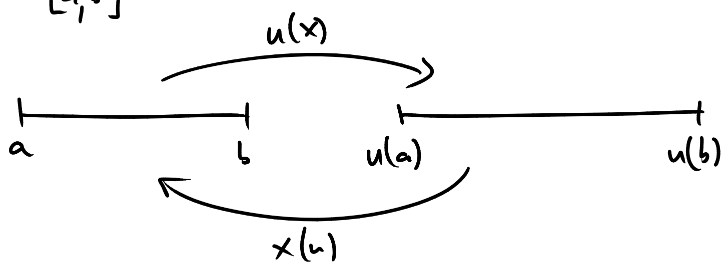
Last time: <u>substitution rule</u>

Formal version:

$\int_a^b f(x)\, dx$

$u = u(x) \longleftrightarrow x = x(u)$

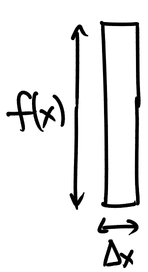
1-1 on $[a,b]$



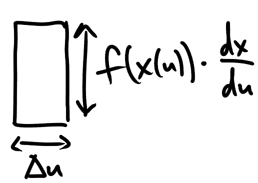
$$\int_a^b f(x)\, dx = \int_{u(a)}^{u(b)} f(x(u)) \cdot \frac{dx}{du}\, du$$

Why does it work?

Think of the integral as coming from a Riemann sum:



vs.



area $= f(x) \cdot \Delta x$

area $= f(x(u)) \cdot \Delta u \cdot \frac{dx}{du}$

but $\Delta u \cdot \frac{dx}{du} \approx \Delta x$,

so area $\approx f(x(u)) \cdot \Delta x$

so the two areas match!

My office hr today: 2-3pm

Ex $\int 2x e^{x^2}\,dx$

$x^2 = u$
$2x\,dx = du$

$= \int e^u\,du$

$= e^u + C$

Ex $\int \tan^2\theta \sec^2\theta\,d\theta$

$u = \tan\theta$
$du = \sec^2\theta\,d\theta$

[$u = \sec\theta\tan\theta$
$du = \dots$
looks complicated!]

$= \int u^2\,du$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}\tan^3\theta + C$

Ex $\int \frac{dx}{1+9x^2}$

$u = 3x$
$du = 3dx$
$\frac{1}{3}du = dx$

$\int \frac{du}{1+u^2} = \tan^{-1}u + C$

$= \int \frac{\frac{1}{3}du}{1+u^2}$

$= \frac{1}{3}\int \frac{du}{1+u^2}$ $\left(= \frac{1}{3}\int \frac{1}{1+u^2}\,du \right)$

$= \frac{1}{3}\tan^{-1}u + C$

$= \frac{1}{3}\tan^{-1}3x + C$

(Check: $\frac{d}{dx}\left(\frac{1}{3}\tan^{-1}3x\right) = \frac{1}{3}\cdot\frac{1}{1+(3x)^2}\cdot 3$
$= \frac{1}{1+9x^2}$ ✓)

Ex $\int \frac{5}{x^2+6x+10}\,dx$

Want to make this look like $\frac{(\text{something})}{u^2+1}$

The trick: "completing the square" — set $u=x+3$ $\quad du=dx$

then $u^2=x^2+6x+9$

so $\int \frac{5}{x^2+6x+10}\,dx = \int \frac{5}{u^2+1}\cdot du$

(in general: for denominator x^2+Ax+B try $u=x+\frac{1}{2}A$)

$= 5\tan^{-1}u + C$

$= 5\tan^{-1}(x+3)+C$

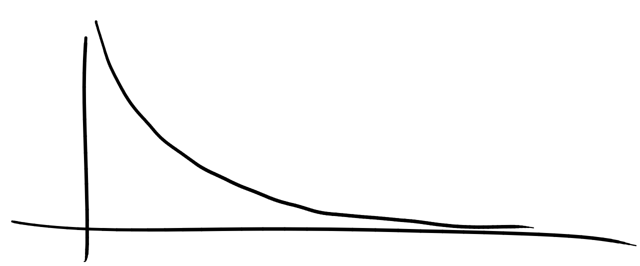
Ex $\int \frac{1}{x\ln x}\,dx \qquad u=\ln x \quad du=\frac{dx}{x}$

$= \int \frac{1}{\ln x}\cdot\frac{dx}{x}$

$= \int \frac{1}{u}\cdot du$

$= \ln|u|$

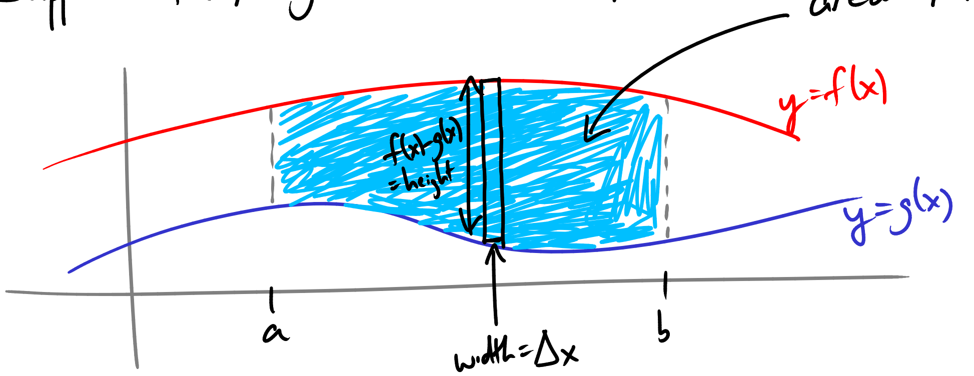
$= \ln|\ln x|$



Areas between curves (Ch 6.1)

Two curves $y=f(x)$ and $y=g(x)$.

Suppose $f(x) > g(x)$ for $x \in [a,b]$.

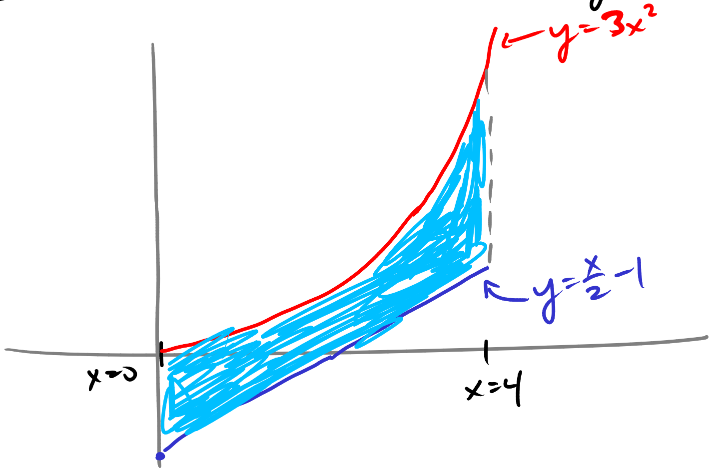


area of rectangle = $\Delta x \left(f(x) - g(x)\right)$

total area = $A = \int_a^b f(x) - g(x)\, dx$

Ex Find the area between the curves $y = 3x^2$ and $y = \frac{x}{2} - 1$ from $x=0$ and $x=4$.

$f(x)$ $g(x)$



$$A = \int_0^4 f(x) - g(x)\, dx$$

$$= \int_0^4 (3x^2) - \left(\frac{x}{2} - 1\right) dx$$

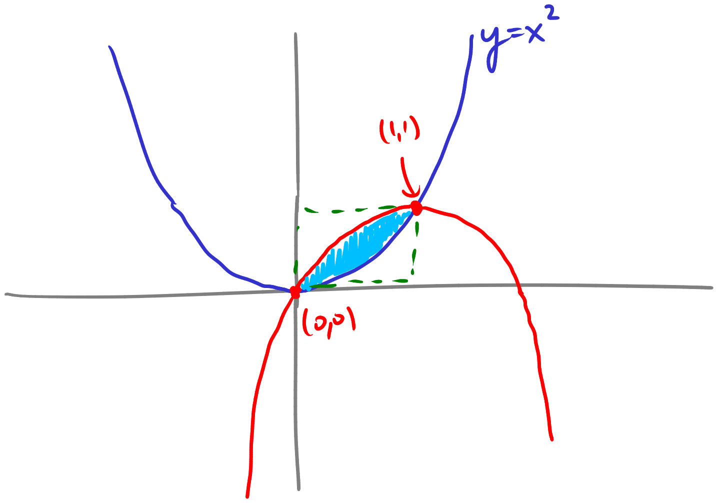
$$= \int_0^4 3x^2 - \frac{x}{2} + 1\, dx$$

$$= x^3 - \frac{x^2}{4} + x \Big|_0^4$$

$$= (64 - 4 + 4) - 0 = \underline{\underline{64}}$$

Ex Find the area of the bounded region between the graphs $y=x^2$ and $y=2x-x^2$.

$y'=2-2x$
$y''=-2$
→ Concave down, max. at $x=1$



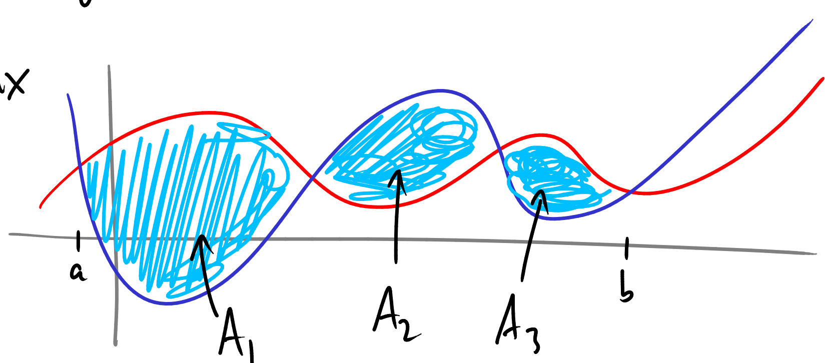
intersection points:

$$x^2=2x-x^2$$
$$2x^2-2x=0$$
$$2x(x-1)=0$$
$$x=0,1$$

$$A=\int_0^1 (2x-x^2)-(x^2)\,dx$$
$$=\int_0^1 2x-2x^2\,dx$$
$$=x^2-\tfrac{2}{3}x^3\Big|_0^1$$
$$=\left(1-\tfrac{2}{3}\right)-(0+0)=\tfrac{1}{3}$$

A rule that finds the area between $y=f(x)$ and $y=g(x)$ no matter which is bigger:

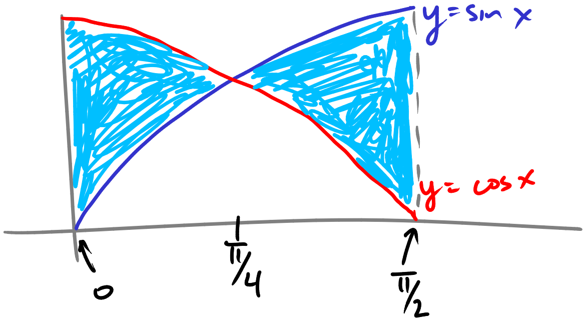
$$A=\int_a^b |f(x)-g(x)|\,dx$$



here $A=A_1+A_2+A_3$

Ex Find the area of the region between $y=\sin x$ and $y=\cos x$ when x ranges between $x=0$ and $x=\frac{\pi}{2}$.

$$A=\int_0^{\pi/2} |\sin x-\cos x|\,dx$$



$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= \sin x + \cos x \Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \left(\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right) + \left((0-1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right)$$

$$= (\sqrt{2} - 1) + (-1 + \sqrt{2})$$

$$= \underline{\underline{2(\sqrt{2}-1)}}$$

Ex Find the area of the region between

$$y = x^3 - x^2 - 7x - 4 = f(x)$$

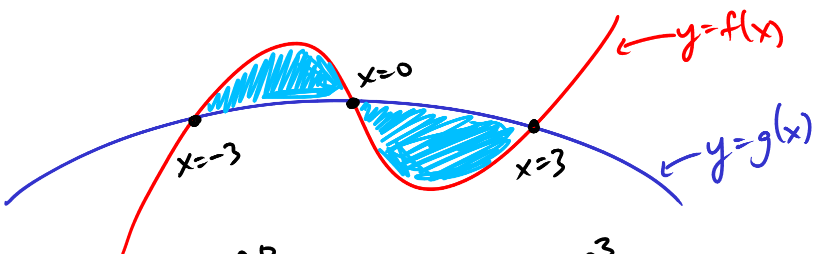
$$y = -x^2 + 2x - 4 = g(x)$$

Points of intersection: $x^3 - x^2 - 7x - 4 = -x^2 + 2x - 4$

$$x^3 - 9x = 0$$

$$x(x^2 - 9) = 0$$

$$x(x+3)(x-3) = 0 \longrightarrow x = 0, 3, -3$$

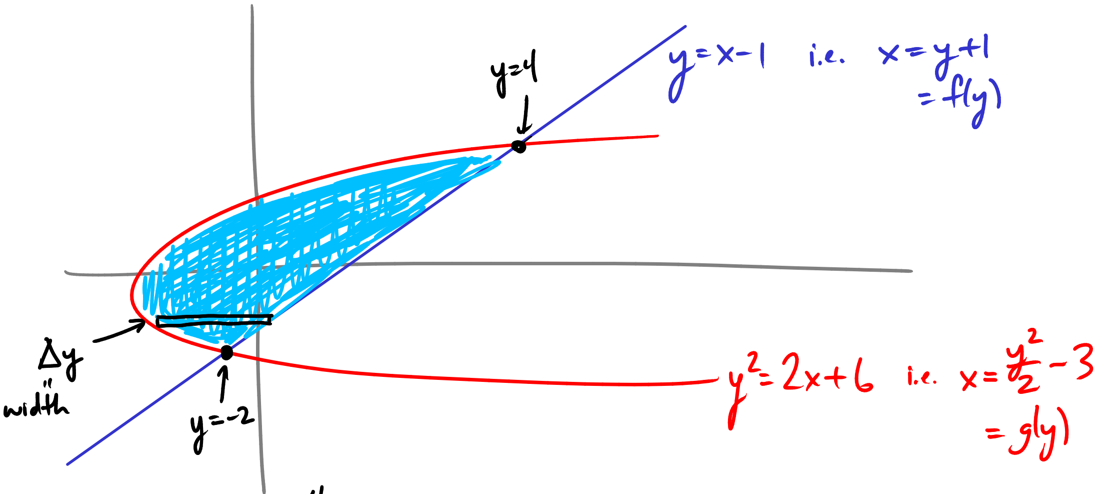


$$A = \int_{-3}^{0} f(x) - g(x)\,dx + \int_0^3 g(x) - f(x)\,dx$$

$$= \cdots = \underline{\underline{\frac{81}{2}}}$$

Ex Find the area between the parabola $y^2 = 2x+6$ $\left(\to \frac{y^2}{2} - 3 = x\right)$

and the line $y = x-1$.



intersections:

$y+1 = \frac{y^2}{2} - 3$

$2y+2 = y^2 - 6$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$y = -2, 4$

$$\text{area} = \int_{-2}^{4} f(y) - g(y)\, dy$$

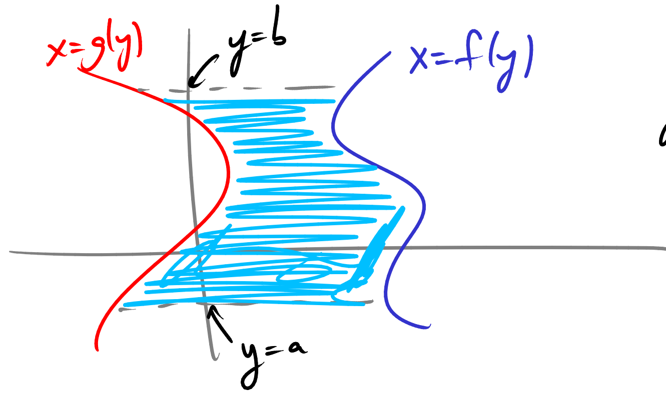
$$= \int_{-2}^{4} (y+1) - \left(\frac{y^2}{2} - 3\right) dy$$

$$= \int_{-2}^{4} -\frac{y^2}{2} + y + 4\, dy$$

$$= -\frac{3}{2}y^3 + \frac{y^2}{2} + 4y \Big|_{-2}^{4}$$

$$= \cdots = \underline{\underline{18}}$$

Generally:



$$\text{area} = \int_a^b f(y) - g(y)\, dy.$$

(<u>if</u> $f(y) > g(y)$ for $y \in [a,b]$)