Lecture 22

17 Nov 2015

Today: my office hr 2-3pm (RLM 9.134)

Last time: substitution (u-substitution)

Ex $\int x e^{-x^2}\,dx$ $\qquad u = x^2 \qquad du = \frac{du}{dx}\cdot dx = 2x\cdot dx$

$\frac{1}{2}du = x\cdot dx$

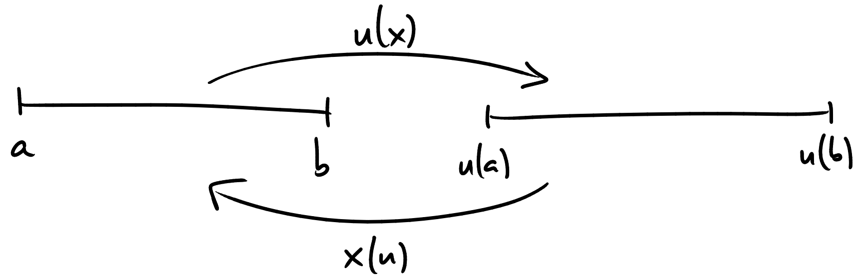
$$= \int e^{-u}\cdot \tfrac{1}{2}\,du$$

$$= -\tfrac{1}{2}e^{-u} + C$$

$$= -\tfrac{1}{2}e^{-x^2} + C$$

Formal version of substitution rule:

$\int_a^b f(x)\,dx$

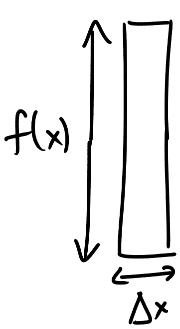


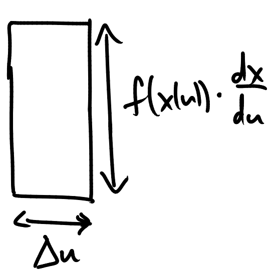
$u(x)$ 1-1 on $[a,b]$

then, $\displaystyle\int_a^b f(x)\,dx = \int_{u(a)}^{u(b)} f(x(u))\cdot\frac{dx}{du}\cdot du$

Why does it work?

Think of the integrals as coming from Riemann sums:





area $= f(x)\cdot\Delta x$ $\qquad$ area $= f(x(u))\cdot\frac{dx}{du}\cdot\Delta u$ $\qquad$ but $\frac{dx}{du}\cdot\Delta u \approx \Delta x$

$\approx f(x(u))\cdot\Delta x$

Ex $\int \tan^2\theta \sec^2\theta \, d\theta$ $\qquad u = \tan\theta \qquad du = \sec^2\theta \, d\theta$

$= \int u^2 \, du$ $\qquad \cancel{u = \sec\theta}$

$\qquad \cancel{u = \tan\theta\sec\theta}$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}(\tan\theta)^3 + C$

(check: $\frac{d}{d\theta}\frac{1}{3}(\tan\theta)^3 = \frac{1}{3}\cdot 3\tan^2\theta \cdot \sec^2\theta = \tan^2\theta\sec^2\theta$ ✓)

Ex $\int \frac{dx}{1+9x^2}$

We know $\int \frac{du}{1+u^2} = \tan^{-1}u$ so try $u = 3x$ $\qquad du = 3\,dx$, $\frac{1}{3}du = dx$

then $\int \frac{dx}{1+9x^2} = \int \frac{\frac{1}{3}du}{1+u^2}$ $\qquad$ (remark: $\int \frac{du}{1+u^2} = \int \frac{1}{1+u^2}du$)

$= \frac{1}{3}\tan^{-1}u + C$

$= \underline{\underline{\frac{1}{3}\tan^{-1}(3x) + C}}$

(Check: $\frac{d}{dx}\left(\frac{1}{3}\tan^{-1}(3x)\right) = \frac{1}{3}\cdot\frac{1}{1+(3x)^2}\cdot 3 = \frac{1}{1+9x^2}$ ✓)

Ex $\int \frac{1}{x^2+4}dx$

$= \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}dx$ $\qquad$ want: $\frac{x^2}{4} = u^2$ — can get this by putting $\frac{x}{2} = u$

$= \frac{1}{4}\int \frac{1}{u^2+1}\cdot 2\,du$ $\qquad u = \frac{x}{2}$, $du = \frac{1}{2}dx \quad 2\,du = dx$

$$= \frac{1}{2}\tan^{-1}(u) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

(Or: take $u = -\frac{x}{2}$, then $\frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx = \frac{1}{4}\int \frac{1}{u^2+1}(-2\,du)$
$-2\,du = dx$

$$= -\frac{1}{2}\tan^{-1}(u) + C$$

$$= -\frac{1}{2}\tan^{-1}\left(-\frac{x}{2}\right) + C$$
)

(These 2 answers are actually the same: $-\frac{1}{2}\tan^{-1}\left(-\frac{x}{2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$
since $\tan^{-1}$ is odd)

Ex $\displaystyle\int \frac{5}{x^2+6x+10}\,dx$ $\qquad \frac{1}{u^2+1}$

Trick: complete the square — let $u = x+3$ $\qquad du = dx$
then $u^2 = x^2+6x+9$

so $\displaystyle\int \frac{5}{x^2+6x+10}\,dx = \int \frac{5}{u^2+1}\,du$

$$= 5\tan^{-1}(u) + C$$

$$= 5\tan^{-1}(x+3) + C$$

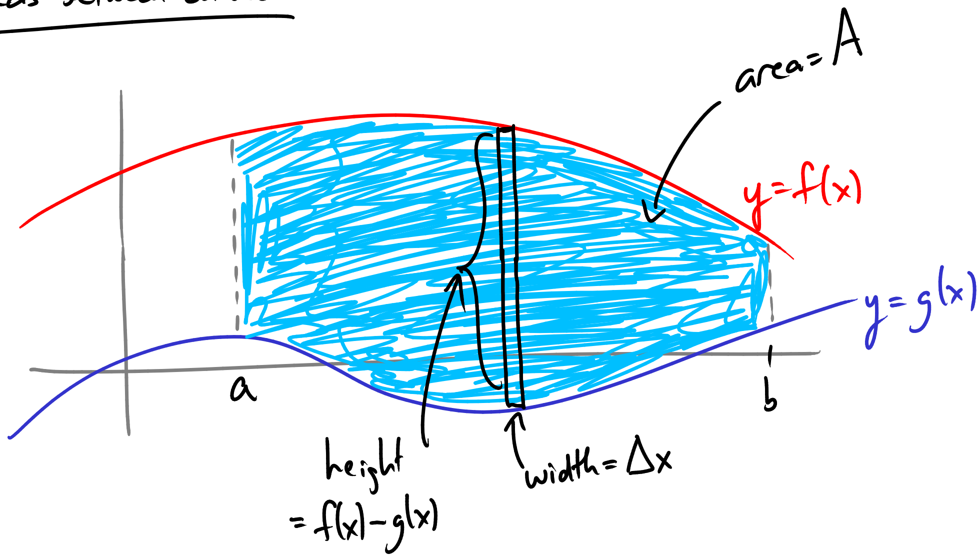
Ex $\displaystyle\int \frac{1}{x\ln x}\,dx$ $\quad (x>0)$ $\qquad u = \ln x$
$du = \frac{dx}{x}$

$$= \int \frac{1}{\ln x}\cdot\frac{dx}{x}$$

$$= \int \frac{1}{u}\cdot du$$

$$= \ln|u| + C = \ln|\ln x| + C$$

Areas between curves (Ch 6.1)

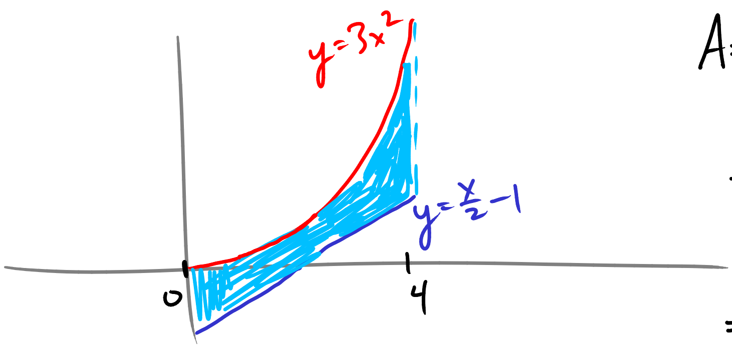


Two curves $y=f(x)$, $y=g(x)$ with $f(x) > g(x)$ for $x \in [a,b]$.

so area of rectangle = $(f(x) - g(x))\,\Delta x$

so total area = $A = \int_a^b (f(x) - g(x))\,dx$

Ex Find the area between the curves $y = 3x^2$ and $y = \frac{x}{2} - 1$ and between $x = 0$ and $x = 4$.



$$A = \int_0^4 (\underbrace{3x^2}_{f(x)}) - (\underbrace{\tfrac{x}{2} - 1}_{g(x)})\,dx$$

$$= \int_0^4 3x^2 - \frac{x}{2} + 1\,dx$$

$$= x^3 - \frac{1}{4}x^2 + x \Big|_0^4$$

$$= \left(4^3 - \frac{1}{4}(4^2) + 4\right) - (0)$$

$$= 64 - 4 + 4 - 0$$

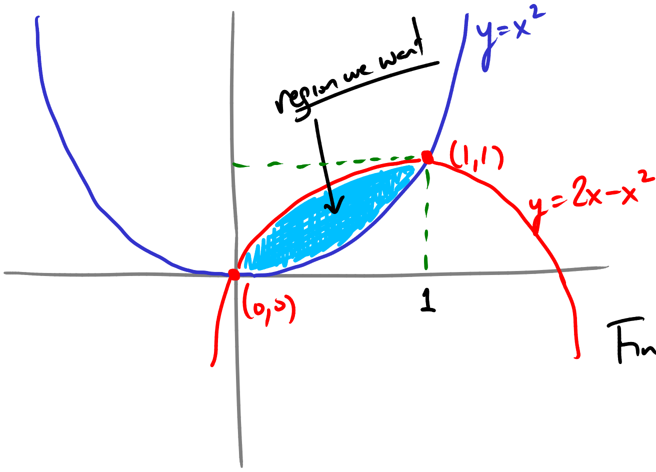
$$= \underline{\underline{64}}$$

Ex Find the area of the bounded region between $y=x^2$ and $y=2x-x^2$.

$y' = 2-2x$

$y'' = -2$

parabola opening down, vertex at $x=1$ $(1,1)$



Find intersection:

$$x^2 = 2x - x^2$$
$$2x^2 = 2x$$
$$2(x^2-x)=0$$
$$2x(x-1)=0$$
$$x=0,1 \longrightarrow \begin{matrix}(0,0)\\(1,1)\end{matrix}$$

$$A = \int_0^1 (2x-x^2) - x^2 \, dx$$

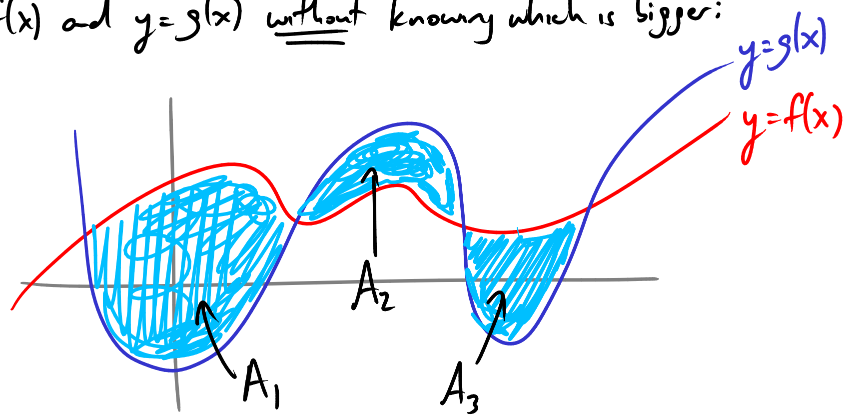
$$= \int_0^1 2x - 2x^2 \, dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1 = \left(1-\frac{2}{3}\right) - 0 = \underline{\underline{\frac{1}{3}}}$$

A rule that finds the area between $y=f(x)$ and $y=g(x)$ <u>without</u> knowing which is bigger:

$$A = \int_a^b |f(x)-g(x)| \, dx$$

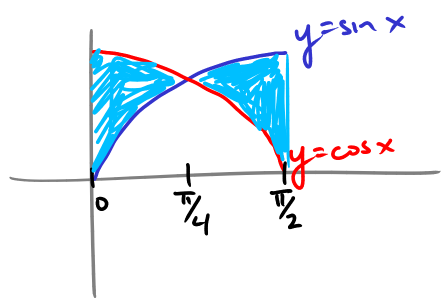
$$= A_1 + A_2 + A_3$$



Ex Find the area of the region bounded by $y=\sin x$ and $y=\cos x$, and $x=0, x=\pi/2$.

$$A = \int_0^{\pi/2} |\sin x - \cos x| \, dx$$

$$= \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$



$$= \sin x + \cos x \Big|_0^{\pi/4} + (-\sin x - \cos x)\Big|_{\pi/4}^{\pi/2}$$

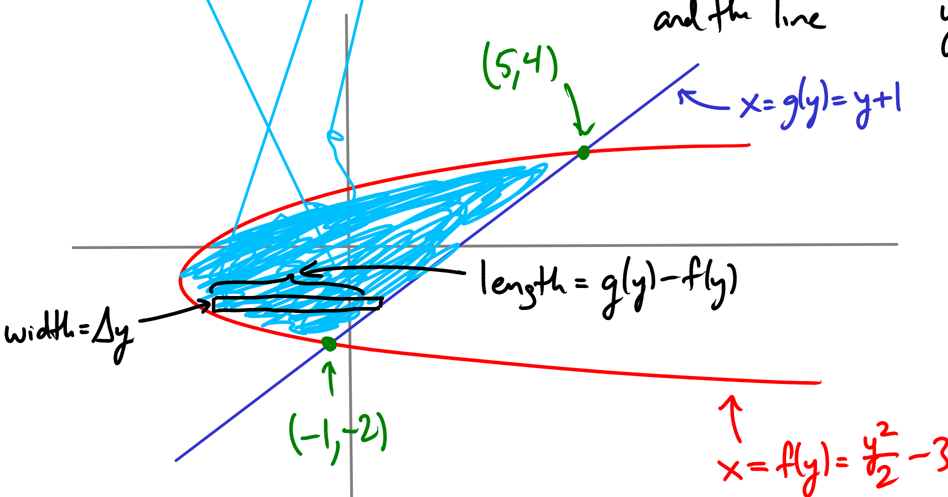
$$= \left(\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right) + \left((-1-0) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right)$$

$$= (\sqrt{2} - 1) + (-1 + \sqrt{2})$$

$$= \underline{\underline{2(\sqrt{2}-1)}}$$

Ex Find the area of the region bounded by the parabola $y^2 = 2x+6$ ie $x = \frac{y^2}{2} - 3$

and the line $y = x - 1$. ie $x = y+1$



intersections:

$$(x-1)^2 = 2x+6$$
$$x^2 - 2x + 1 = 2x + 6$$
$$x^2 - 4x - 5 = 0$$
$$(x-5)(x+1) = 0$$
$$x = -1, 5$$

Chop the region into <u>horizontal</u> rectangles

each rectangle has area $= (g(y) - f(y)) \cdot \Delta y$

total area $=$ $A = \displaystyle\int_{-2}^{4} (g(y) - f(y))\, dy$

$$= \int_{-2}^{4} (y+1) - \left(\frac{y^2}{2} - 3\right) dy = \cdots = \underline{\underline{18}}$$