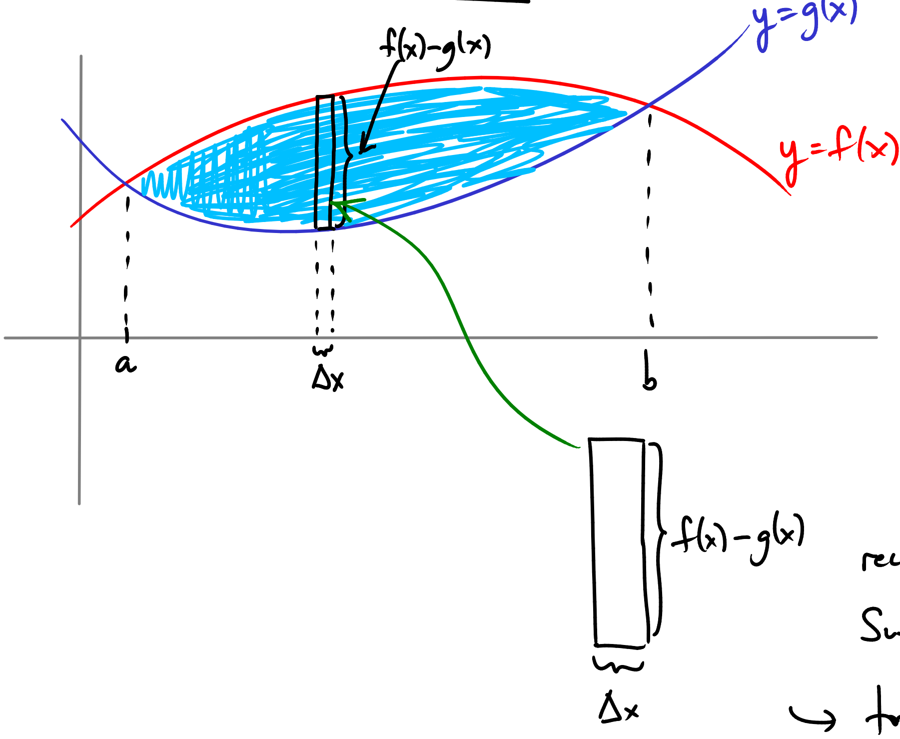
Lecture 23

19 Nov 2015

Last time: area between curves



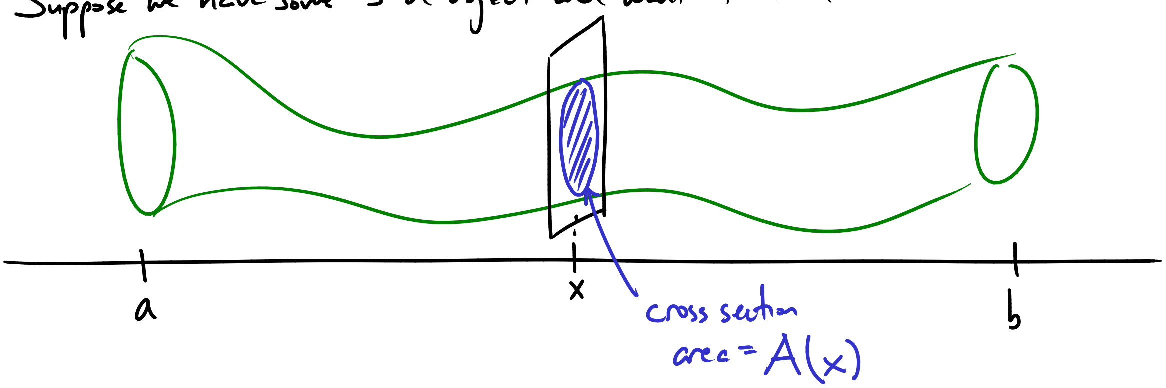
rectangle area $= \Delta x \left(f(x)-g(x)\right)$

Sum them all up:

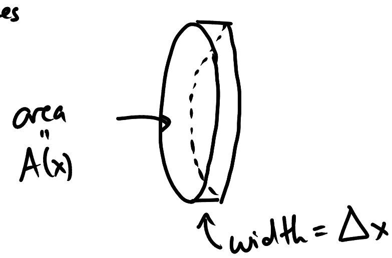
$\hookrightarrow$ total area $= \int_a^b dx \left(f(x)-g(x)\right)$

Volume (Ch 6.2)

Suppose we have some 3-d object and want to find its volume.



Chop the object into slices which look like "pancakes"

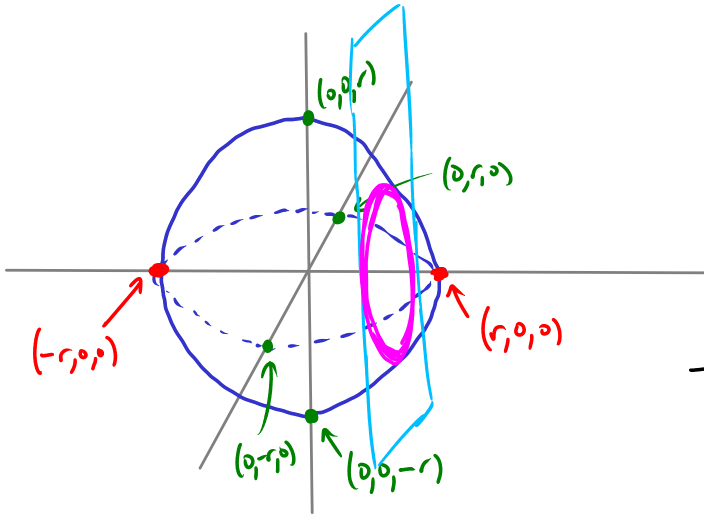


Volume of each slice:

$$V = A(x)\cdot\Delta x$$

To get the total volume of our object, add up the slices: $V = \int_a^b A(x)\,dx$

Ex Calculate the volume of a sphere of radius r.



Slice the sphere by planes

$x = \text{constant}.$

Sphere is $x^2+y^2+z^2 \leq r^2$

At fixed value of x:

this is $y^2+z^2 \leq r^2-x^2$

This is the inside of a circle, with radius $= \sqrt{r^2-x^2}$.

So the cross sections are circles, with area $A(x) = \pi\left(\sqrt{r^2-x^2}\right)^2$

ie $A(x) = \pi(r^2-x^2)$

Volume of sphere: $V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx$

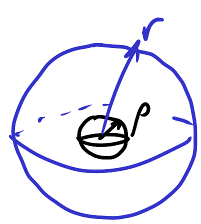
$= \pi\left(r^2x - \frac{1}{3}x^3\right)\Big|_{-r}^{r}$

$= \pi \cdot \left(\left(r^3 - \frac{1}{3}r^3\right) - \left(-r^3 + \frac{1}{3}r^3\right)\right)$

$= \pi\left(\frac{2}{3}r^3 + \frac{2}{3}r^3\right) = \pi\frac{4}{3}r^3$

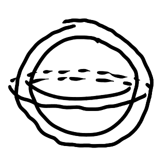
Remark: another way to get this volume —

chop up the sphere into concentric shells



shell of radius ρ has surface area $4\pi\rho^2$ $\quad 0 \leq \rho \leq r$

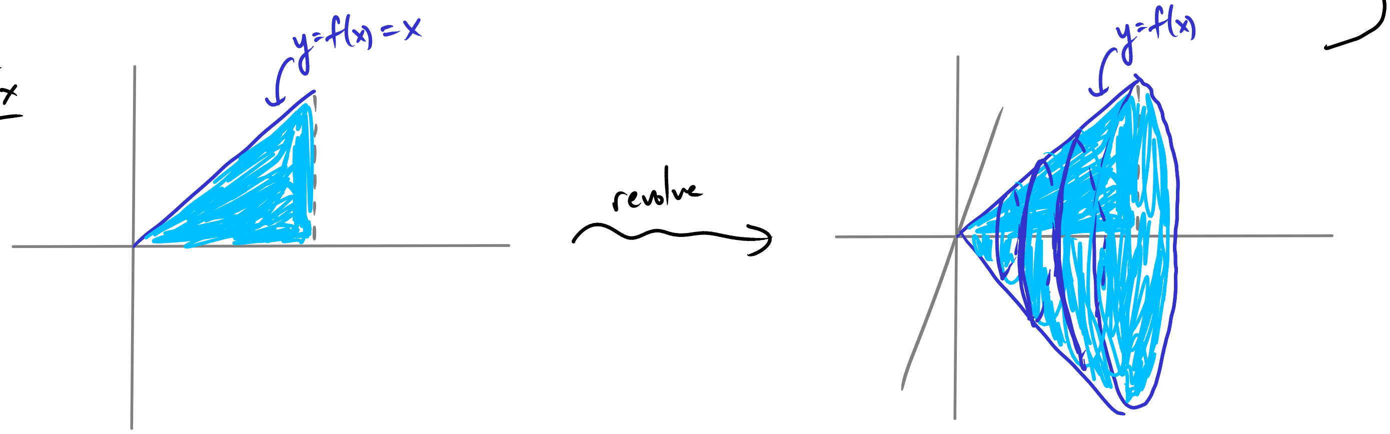
volume of shell $= 4\pi\rho^2 \cdot \Delta\rho$

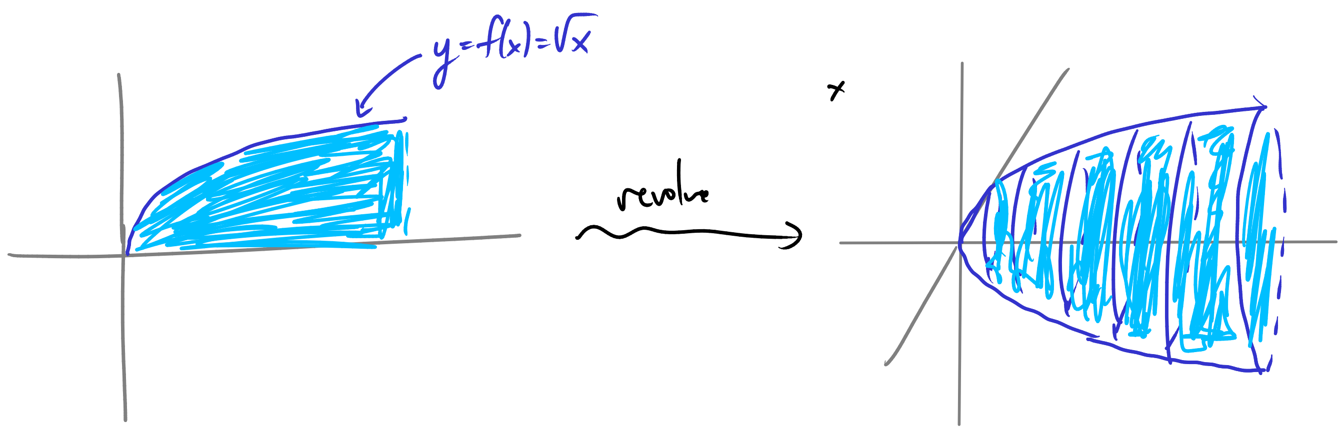


$\longrightarrow$ total volume $= \int_0^r 4\pi\rho^2\,d\rho = \frac{4}{3}\pi r^3$

A common type of solid: "solid of revolution" — take some region in the x-y plane and <u>revolve</u> it around, say, the x-axis.

<u>Ex</u>





The cross section of such a solid is a <u>circle</u> of radius $f(x)$.

So cross section <u>area</u> is $A(x) = \pi \cdot f(x)^2$.

Can use this to get the volume.

<u>Ex</u> Find the volume of the solid obtained by revolving the area between $y=\sqrt{x}$ and x-axis, <u>around</u> the x-axis, with x running from 0 to 2.

$$V = \int_0^2 dx\, A(x) = \int_0^2 \pi \cdot (\sqrt{x})^2\, dx = \int_0^2 \pi \cdot x\, dx = \pi \cdot \frac{x^2}{2}\Big|_0^2 = \underline{\underline{2\pi}}$$

Can also revolve around, say, the y-axis.

<u>Ex</u> Find the volume of the solid obtained by revolving the region bounded by

$x = y - y^2$

$x = 0$

around the y-axis.

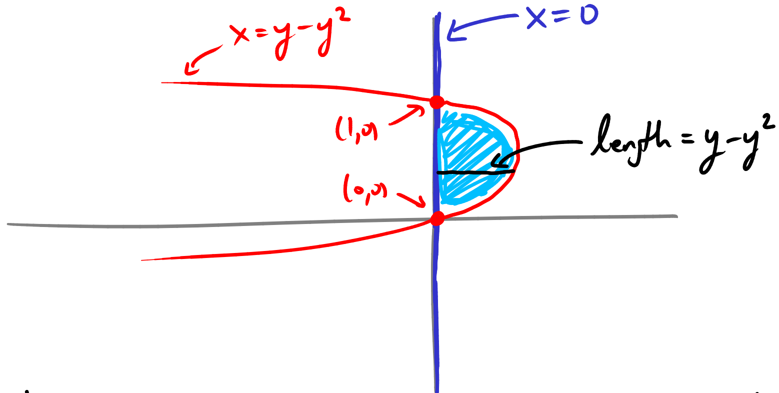
Points of intersection:

$y - y^2 = 0$

$y(y-1) = 0$

$y=0$ or $y=1$

$(0,0)$ $(0,1)$



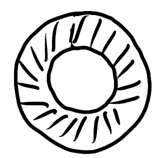
Slice by "horizontal" planes, $y=\text{constant}$: cross sections are circles of radius $= y-y^2$

$$A(y) = \pi(y-y^2)^2$$

$$V = \int_0^1 dy\, A(y) = \int_0^1 \pi\,(y-y^2)^2\, dy$$

$$= \cdots = \underline{\underline{\frac{\pi}{30}}}$$

Another common shape: cross sections which are "washers"

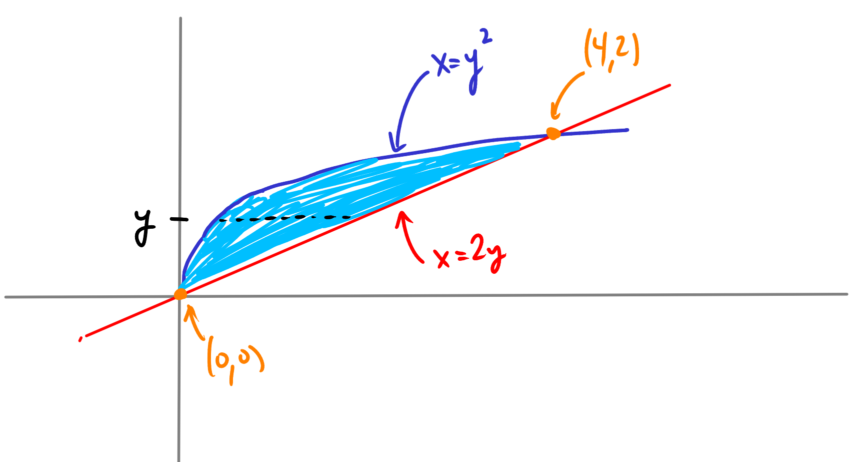


Ex Let R be the region between $y=\sqrt{x}$ and $x=2y$.

Find the volume of the solid obtained by rotating R around the y-axis.

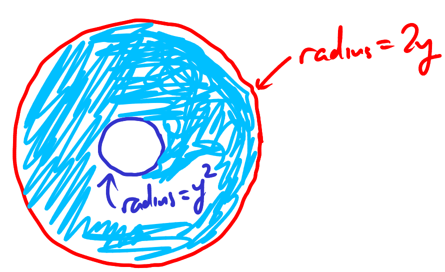
Intersections:
$$\left.\begin{array}{l} y=\sqrt{x} \longrightarrow x=y^2 \\ x=2y \end{array}\right\} \longrightarrow 2y=y^2,\ y^2-2y=0,\ y(y-2)=0,\ y=0 \text{ or } y=2$$

$(0,0)$ $(4,2)$



Slice by planes $y=\text{constant}$.

Cross section:



Cross section area:

$$A(y) = \pi(2y)^2 - \pi\left(y^2\right)^2$$

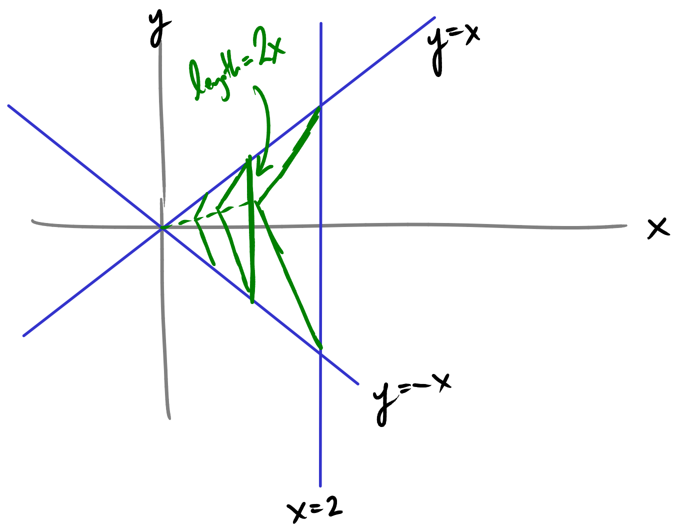
Volume: $\int_0^2 A(y)\, dy$

$$= \int_0^2 \pi(2y)^2 - \pi(y^2)^2 \, dy$$

$$= \pi \int_0^2 4y^2 - y^4 \, dy$$

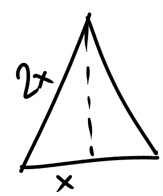
$$= \dots = \underline{\underline{\frac{64}{15}\pi}}$$

<u>Ex</u> Calculate the volume of a solid whose <u>base</u> is the region between $y=x$, $y=-x$ and $x=2$ and whose cross sections at fixed x are equilateral Δ's.



$$V = \int_0^2 A(x) \, dx$$

$A(x)$ = area of equilateral Δ with side length $= 2x$



$$\longrightarrow A(x) = \sqrt{3}\,x^2$$

$$V = \int_0^2 A(x) \, dx = \int_0^2 \sqrt{3} \cdot x^2 \, dx = \dots = \underline{\underline{\frac{8}{3}\sqrt{3}}}$$