Lecture 24

24 Nov 2015

HW14 (last one) due next Fri

Midterm 3 next Tue — covers thru Lecture 22 / HW13

My office hr: today as usual

extra Monday 2:30–3:30

Last time: computing volumes by integration

Average values

What do we mean by the average of some function f?

e.g. "average temperature over a day" — $f(t)$ = temperature at time t

What do we do to f to get the average?

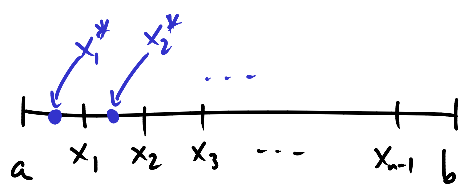
Average of a finite collection of numbers:

average of $\{2,4\}$ is $\frac{2+4}{2} = \frac{6}{2} = 3$

average of $\{2,4,7\}$ is $\frac{2+4+7}{3} = \frac{13}{3}$

average of $\{y_1, y_2, \dots, y_n\}$ is $\frac{y_1+y_2+y_3+\cdots+y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} y_i$

To define average of a function $f(x)$ on the domain $[a,b]$:



take the average of the sample values — $y_i = f(x_i^*)$

the approximate average is $\frac{y_1+\cdots+y_n}{n} = \frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{n} = \sum_{i=1}^{n} f(x_i^*)\cdot\left(\frac{1}{n}\right)$

Looks like a Riemann sum! If we wanted to calculate

$\int_a^b f(x)\,dx$, we would write Riem sum $\sum_{i=1}^{n} f(x_i^*)\cdot \Delta x$

$$= \sum_{i=1}^{n} f(x_i^*)\cdot\left(\frac{b-a}{n}\right)$$

Comparing those two: $\int_a^b f(x)\,dx = (b-a)\left(\begin{array}{c}\text{average value of } f(x)\\ \text{on interval } [a,b]\end{array}\right)$

i.e.

The average value of $f(x)$ on the interval $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

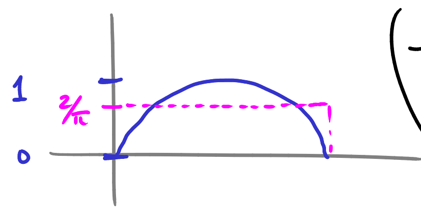
Ex The average value of $f(x) = \sin x$ over $[0,\pi]$ is

$$\frac{1}{\pi - 0}\int_0^\pi \sin x\,dx$$

$$= \frac{1}{\pi}\left(-\cos x \Big|_0^\pi\right)$$

$$= \frac{1}{\pi}\left(-(-1) - (-1)\right)$$

$$= \frac{1}{\pi}(2) = \underline{\underline{\frac{2}{\pi}}}$$



(the rectangle has same area as the area under sine graph)

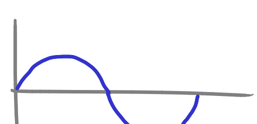
Ex The average value of $f(x) = c$ (c constant) over $[a,b]$ is

$$\frac{1}{b-a}\int_a^b c\;dx = \frac{1}{b-a}\cdot\left(cx\Big|_a^b\right)$$

$$= \frac{1}{b-a}\cdot c(b-a)$$

$$= \underline{\underline{c}}$$ (as we should expect).

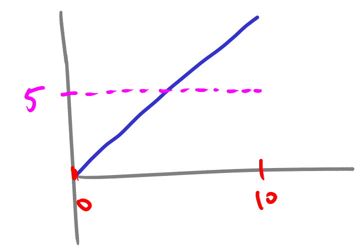
Ex The average value of $\sin(x)$ over $[0,2\pi]$ is



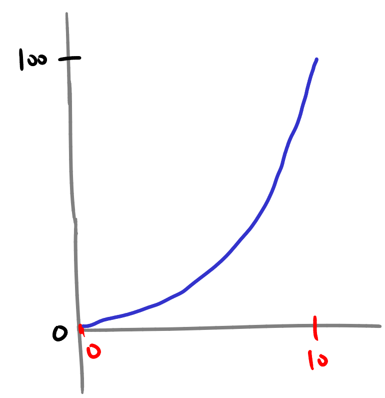
$$\frac{1}{2\pi - 0}\int_0^{2\pi} \sin(x)\,dx = \cdots = \underline{\underline{0}}.$$

Ex The average value of $f(x) = x$ over $[0,10]$ is

$$\frac{1}{10-0}\int_0^{10} x\,dx = \frac{1}{10}\left(\frac{x^2}{2}\Big|_0^{10}\right) = \frac{1}{10}\left(\frac{100}{2}\right) = \frac{1}{10}(50) = \underline{\underline{5}}$$



Ex The average value of $f(x) = x^2$ over $[0,10]$ is



$$\frac{1}{10-0}\int_0^{10} x^2\,dx$$

$$= \frac{1}{10}\left(\frac{x^3}{3}\Big|_0^{10}\right) = \frac{100}{3} \approx 33.3$$

Ex The average value of $f(x) = \sin^2 x$ over $[0, 2\pi]$ is..

First method: $\displaystyle\frac{1}{2\pi - 0}\int_0^{2\pi} \sin^2 x\,dx$

Use trig identity:

$$\cos 2x = 2\sin^2 x - 1$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

so have $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2x)\,dx$

$$= \frac{1}{4\pi}\int_0^{2\pi} 1 - \cos 2x\,dx$$

try $u = \sin x$

$du = \cos x\,dx \rightsquigarrow \displaystyle\frac{1}{2\pi}\int u^2 \frac{du}{\cos x}$

$dx = \dfrac{du}{\cos x}$

$$= \frac{1}{2\pi}\int u^2 \frac{du}{\sqrt{1-u^2}}$$

$u^2 = \sin^2 x$

$1 - u^2 = 1 - \sin^2 x = \cos^2 x$

$\sqrt{1-u^2} = \cos x$

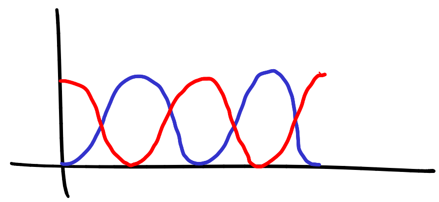
$\longrightarrow$ No HELP

$$= \frac{1}{4\pi}\left(x - \frac{1}{2}\sin 2x \right|_0^{2\pi}\Big)$$

$$= \frac{1}{4\pi}\left(\left(2\pi - \frac{1}{2}(0)\right) - \left(0 - \frac{1}{2}(0)\right)\right)$$

$$= \frac{2\pi}{4\pi} = \underline{\underline{\frac{1}{2}}}$$

<u>Second method:</u> Notice that the average of $\sin^2 x$ and $\cos^2 x$ over $[0, 2\pi]$ must be the <u>same</u>



and $\sin^2 x + \cos^2 x = 1$

so the averages must add up to 1

so each must have average $\underline{\underline{\frac{1}{2}}}$.

<u>Work</u>

A definition from physics:

if an object moves for a distance Δx,

acted on by a <u>constant</u> force F ($F > 0$ for force pushing in <u>positive</u> dir,
$F < 0$ for force " " <u>negative</u> dir)

then we say the force does <u>work</u> on the object,

$$W = F \cdot \Delta x$$

<u>Ex</u> To lift a rock weighing 1 kg
for a height $\Delta x = \frac{1}{2}$ m
(with constant speed)

we have to exert a force $F = (1\,\text{kg}) \cdot (\underbrace{9.8\ \text{m/s}^2}_{\text{gravity}}) = 9.8\ \text{kg} \cdot \text{m/s}^2$

so the <u>work</u> we have to do is $W = F \cdot \Delta x$

$$= (9.8\ \text{kg} \cdot \text{m/s}^2) \cdot \left(\frac{1}{2}\ \text{m}\right) = 4.9\ \text{kg} \cdot \frac{\text{m}^2}{\text{s}^2}$$

$$= \underline{\underline{4.9\ \text{J}}}$$

What if the force is not constant?

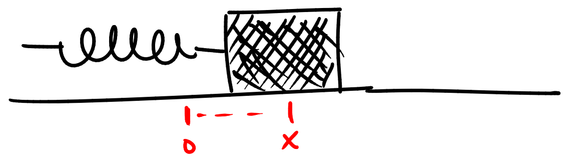
Doesn't make sense to write $W = F \cdot \Delta x$

Instead, $W = \int F \cdot dx$

(One way of thinking about this: break the process up into many sub-processes,

$$W = \sum F(x_i^*)\Delta x, \longrightarrow \int F\,dx \;)$$

Ex A block is attached to a spring



When the block is at position x

the spring exerts a force

$$-kx$$

(k = "spring constant")

If $k = 2\,\frac{N}{m}$, what is the work done by the spring on the block as it moves from $x = 0\,m$ to $x = .03\,m$?

$$W = \int_0^{.03} F\,dx = \int_0^{.03} -2x\,dx = -x^2\Big|_0^{.03} = -\left(\frac{3}{100}\right)^2$$

$$= -\frac{9}{10000}$$

$$= -.0009\ J$$

Why do we want to calculate the work?

$$F = m\,a$$ (mass, accel)

(F = net force)

Because:

Total work

$$W = \int_{x_0}^{x_1} F\,dx$$

$$= \int_{x_0}^{x_1} m a\,dx$$

$$= \int_{x_0}^{x_1} m\frac{dv}{dt}\,dx$$

$$= \int_{v_0}^{v_1} m \frac{dx}{dt} dv$$

kinetic energy

$$= \int_{v_0}^{v_1} m v \, dv$$

$$= \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 \quad = \quad \underline{\text{net change}} \text{ in } \frac{1}{2}mv^2$$