Lecture 24

24 Nov 2015

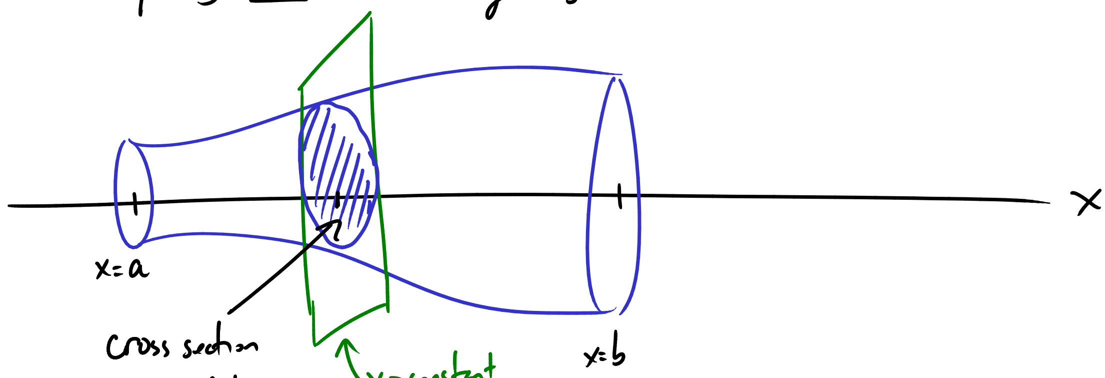
HW14 (last one) due next Friday

Midterm 3 next Tue — covers up to Lecture 22/HW13

My office hr: today as usual 4–5:30

extra next Monday 2:30–3:30

Last time: computing volumes of solids by integration

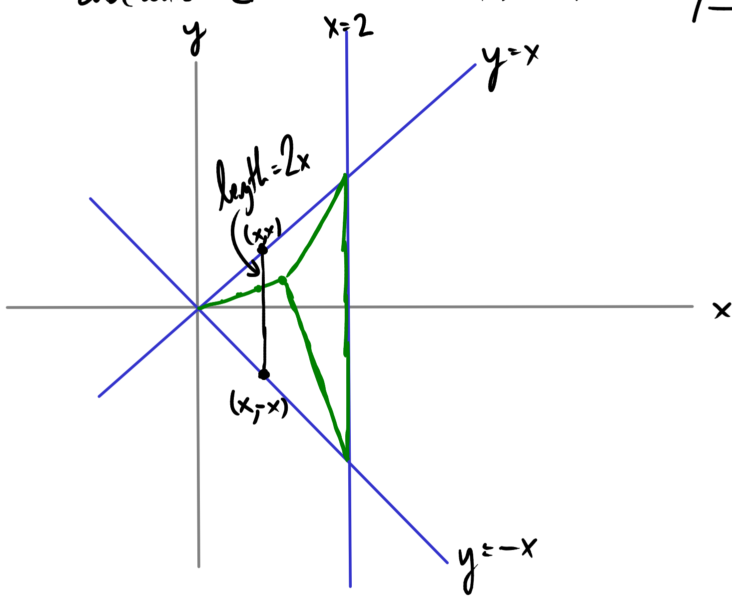


$$\text{total volume} = \int_a^b A(x)\,dx$$

In examples from last time, cross sections were discs or washers — all made from circles

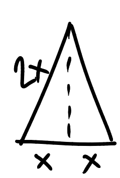
One example that's different:

Ex Calculate the volume of a solid whose base is the region between $y=x$, $y=-x$ and $x=2$ and whose cross sections at fixed x are equilateral triangles.



$$V = \int_0^2 A(x)\,dx$$

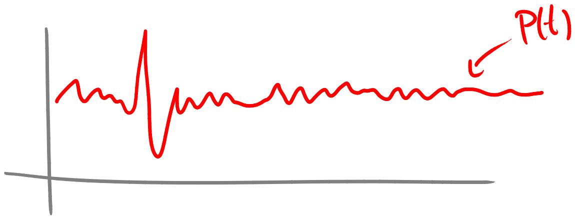
$A(x)$ = area of equilateral triangle with side length $2x$



$\rightarrow A(x) = \sqrt{3}\,x^2$

So $V = \int_0^2 \sqrt{3}x^2\,dx = \cdots = \underline{\underline{\frac{8}{3}\sqrt{3}}}$

Average values



$P(t)$ = price of coffee at time t

How do we define the average price of coffee over the last year?

We know how to take average of a finite collection of numbers:

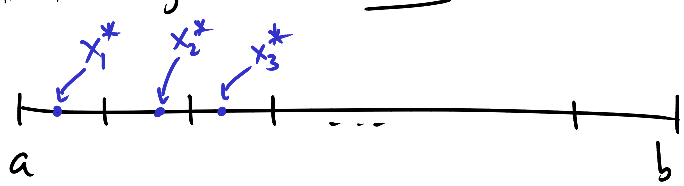
average of $\{2,4\} = \frac{2+4}{2} = \frac{6}{2} = 3$

" " $\{1,1,1,1,7\} = \frac{1+1+1+1+7}{5} = \frac{11}{5}$

" " $\{-1,0,2\} = \frac{-1+0+2}{3} = \frac{1}{3}$

" " $\{y_1, y_2, \dots, y_n\} = \frac{y_1+y_2+\dots+y_n}{n} = \sum_{i=1}^{n} y_i \cdot \frac{1}{n}$

So, to define the average value of a function $f(x)$ on domain $[a,b]$:



take sample values $y_i = f(x_i^*)$

average these: $\frac{y_1+y_2+\dots+y_n}{n} = \sum_{i=1}^{n} y_i \cdot \frac{1}{n} = \sum_{i=1}^{n} f(x_i^*) \cdot \frac{1}{n}$

This looks like a Rieman sum!

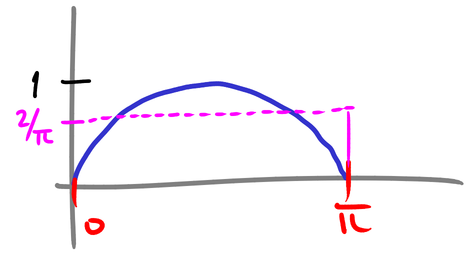
Riemann sum for $\int_a^b f(x)\,dx$: $\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x = \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n}$

What this means: $\begin{pmatrix}\text{approximate}\\ \text{average of}\\ f \text{ over}\\ [a,b]\end{pmatrix} = \frac{1}{b-a} \cdot \begin{pmatrix}\text{approximate}\\ \int_a^b f(x)\,dx\end{pmatrix}$

So, we define:

The average value of $f(x)$ on the interval $[a,b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Ex Compute the average value of $f(x) = \sin x$ on $[0,\pi]$.

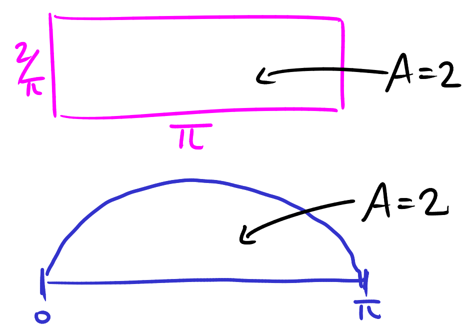


$$A = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi}\left(-\cos x \Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(-(-1) - (-1)\right)$$

$$= \underline{\underline{\frac{2}{\pi}}}$$



(area of rectangle = area under sin x)

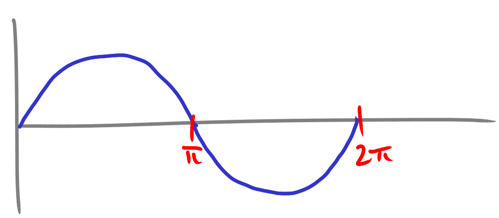
Ex The average value of $f(x) = c$ (c constant) on $[a,b]$ is:

$$\frac{1}{b-a}\int_a^b c\,dx = \frac{1}{b-a}\left(cx\Big|_a^b\right)$$

$$= \frac{1}{b-a}\cdot c(b-a)$$

$$= \underline{\underline{c}} \qquad \text{(as expected)}$$

Ex The average value of $\sin(x)$ over $[0, 2\pi]$ is



$$\frac{1}{2\pi - 0}\int_0^{2\pi} \sin x\,dx$$

$$= \frac{1}{2\pi}\left(-\cos x\Big|_0^{2\pi}\right)$$

$$= \frac{1}{2\pi}\left(-1 - (-1)\right) = \underline{\underline{0}}$$

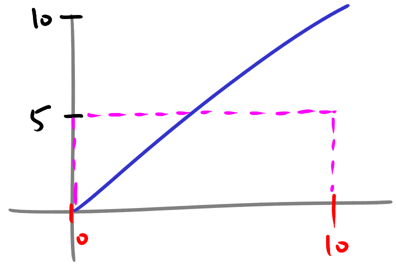
Ex The average value of $f(x) = x$ over $[0,10]$ is

$$\frac{1}{10-0}\int_0^{10} x\,dx$$

$$= \frac{1}{10}\left(\frac{x^2}{2}\Big|_0^{10}\right)$$

$$= \frac{1}{10}\left(\frac{100}{2} - 0\right)$$

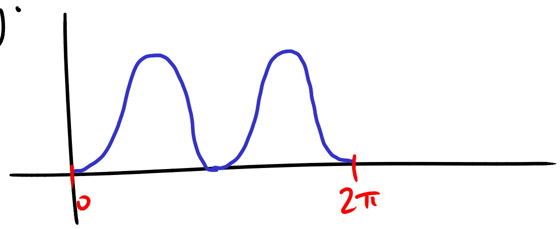
$$= \frac{50}{10} = \underline{5}$$



as you would guess.

Ex The average value of $f(x) = \sin^2 x$ over $[0, 2\pi]$ is

Tricky way:



$$\sin^2 x + \cos^2 x = 1$$

so (average of $\sin^2 x$) + (average of $\cos^2 x$) $= 1$

but average of $\sin^2 x$ = average of $\cos^2 x$

so both must equal $\frac{1}{2}$. So the average is $\frac{1}{2}$.

Direct way:

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\,dx = ?$$

Use trig identity: $\cos 2x = 2\sin^2 x - 1$

$$\sin^2 x = \frac{1}{2}(\cos 2x + 1)$$

so have $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\cos 2x + 1)\,dx$

$$= \cdots = \frac{1}{2\pi}(\pi) = \underline{\frac{1}{2}}.$$

$u = \sin x$

$du = \cos x\,dx$

$\frac{du}{\cos x} = dx$

$\cos x = \sqrt{1-u^2}$

$$\rightarrow \int u^2 \frac{du}{\cos x}$$

$$= \int u^2 \frac{du}{\sqrt{1-u^2}}$$

NO HELP

Work

A definition from physics:

if an object moves for a distance Δx,
acted on by a constant force F

($F>0$ for forces pushing in +ve direction
$F<0$ " " " -ve ")

we say the force does work on the object,

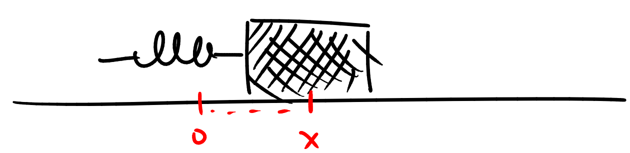
$$W = F \cdot \Delta x$$

Ex to lift a rock weighing 1 kg
for a height $\Delta x = \frac{1}{2} m$
with constant speed,
need a force $F = (1\,kg)\cdot(9.8\,m/s^2) = 9.8\,N$
the work is $W = F \cdot \Delta x = (9.8\,N)\cdot(\frac{1}{2}\,m) = \underline{\underline{4.9\ J}}$

What if the force F is not constant?
Then we put

$$W = \int F\,dx$$

Ex A block is attached to a spring



When the block is at position x
the spring exerts a force

$$F = -kx$$

$k =$ "spring constant"

When the block moves from $x=0$ to $x=c$, how much work does the spring do on the block?

$$W = \int_0^c F\,dx = \int_0^c (-kx)\,dx = -\frac{1}{2}kx^2 \Big|_0^c = \underline{\underline{-\frac{1}{2}kc^2}}$$

Why do we want to calculate the work?

Recall: if F is the total (net) force on an object then $F = ma$ (acceleration)

Then the total work over some process is (initial pos x_0, final pos x_1)

$$W = \int_{x_0}^{x_1} F\,dx$$

$$= \int_{x_0}^{x_1} ma\,dx$$

$$= \int_{x_0}^{x_1} m\frac{dv}{dt}\,dx$$

$$= \int_{v_0}^{v_1} m\frac{dx}{dt}\,dv$$

$$= \int_{v_0}^{v_1} mv\,dv = \frac{1}{2}mv^2\Big|_{v_0}^{v_1} = \underline{\underline{\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2}}$$

change in kinetic energy!