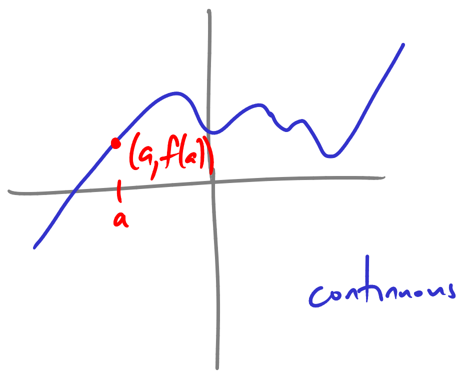
Lecture 6 | 12 Sep 2018

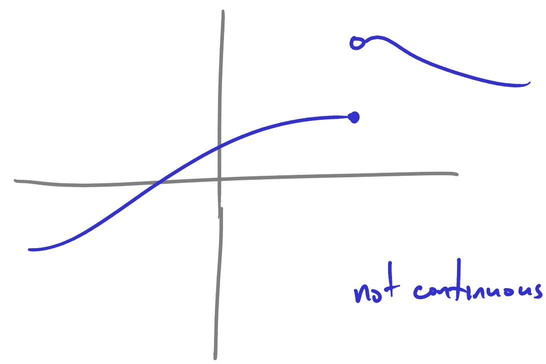
My office hr W 4-5 ← today
M 2-3 RLM 9.134

Last time: limits

Continuity

Informally: we say f is continuous if "we can draw its graph without lifting the pencil"

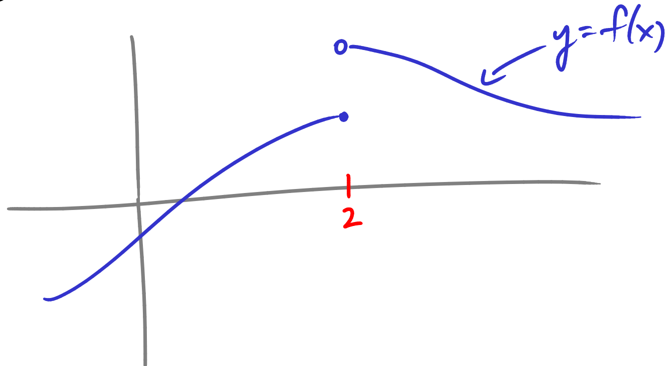




Formally: we say f is continuous at a if $\lim\limits_{x \to a} f(x) = f(a)$.

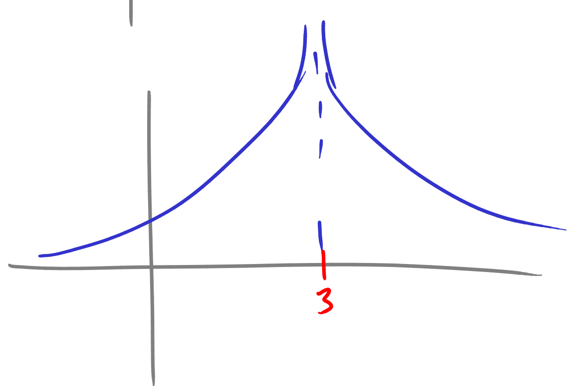
[ie if ① a is in the domain of f, ie $f(a)$ exists,
② $\lim\limits_{x \to a} f(x)$ exists,
③ $\lim\limits_{x \to a} f(x) = f(a)$.]

Ex



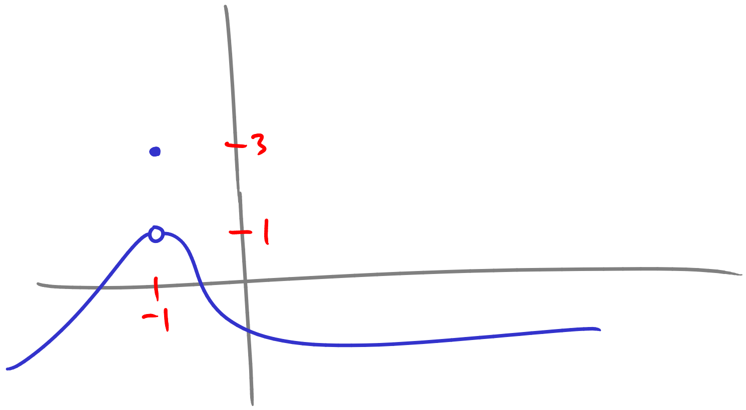
this $f(x)$ is continuous (cts) at all a except $a=2$.
(fails condition ②) "jump discontinuity"

Ex



this $f(x)$ is cts at all a except $a=3$.
(fails condition ①: 3 is not in domain)

Ex

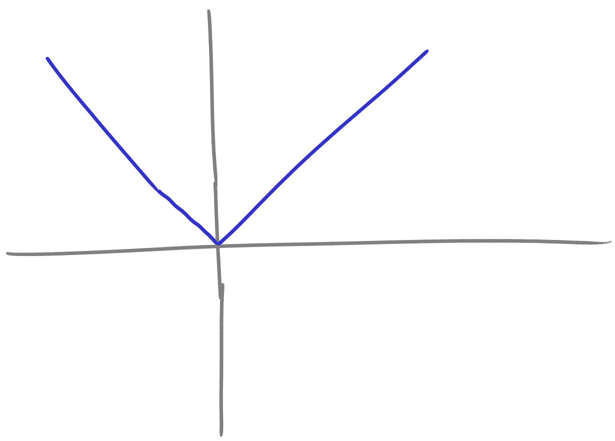


$f(x)$ is cts everywhere except $a=-1$.

fails condition ③:

$$f(-1) \neq \lim_{x\to -1} f(x)$$
$$\begin{array}{ccc} \shortparallel & & \shortparallel \\ 3 & & 1 \end{array}$$

Ex



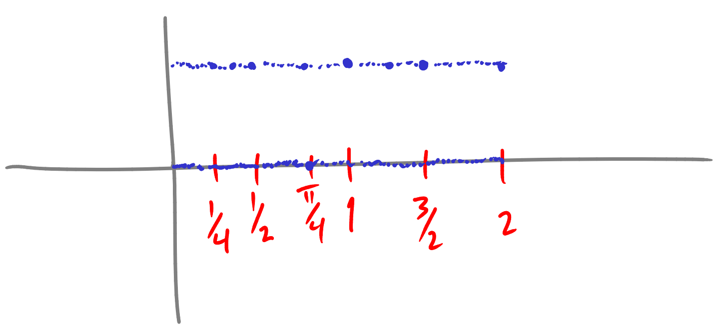
$f(x) = |x|$

cts at <u>all</u> a

to check $a=0$:

$$f(0) = \lim_{x\to 0} |x| \checkmark$$
$$f(0) = |0| = 0, \qquad \lim_{x\to 0}|x| = 0$$

Ex $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational } (x = \frac{p}{q} \quad p,q \text{ integers}) \\ 0 & \text{if } x \text{ is irrational} \end{cases}$



$f(x)$ is cts at <u>no</u> value of a.

Fact If f, g are cts at a then

$f+g$ is cts at a

$f-g$ " " " "

cf " " " " (c = any constant)

fg " " " "

f/g " " " " (if $g(a) \neq 0$)

Fact The following are continuous everywhere in their domain:

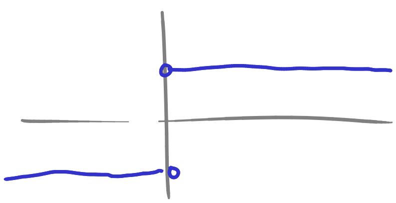
polynomials, rational functions, roots, trig functions, inverse trig functions, exp, log, absolute value, composition of cts functions

Ex $f(x) = x^2 + 3x - 104$ is cts at all a.

$f(x) = \dfrac{x^2-4}{x+1} + \cos(x^3)$ is cts at all $a \neq -1$.

$f(x) = \sin^2\left(\dfrac{x+3}{x^2-4}\right)$ is cts at all $a \neq \pm 2$.

$f(x) = \dfrac{|x|}{x}$ is cts at all $x \neq 0$.



Ex $\lim\limits_{x\to 5} \sin\left(\dfrac{x+4}{x-7}\right) = \sin\left(\dfrac{5+4}{5-7}\right) = \sin\left(-\dfrac{9}{2}\right) = \underline{\underline{-\sin\left(\dfrac{9}{2}\right)}}$

because $\sin\left(\dfrac{x+4}{x-7}\right)$ is cts.

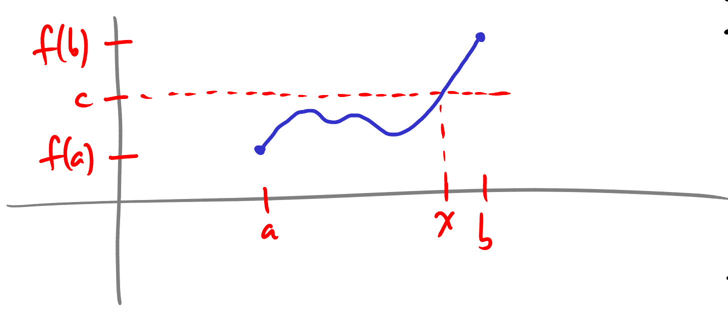
What about: $\lim\limits_{x\to 0} \cos^{-1}\left(\dfrac{\sin x}{x}\right)$?

Fact: limits "pass through" continuous functions — if f is cts at b, and $\lim\limits_{x\to a} g(x) = b$,

then $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right) = f(b)$.

So here, $\lim\limits_{x\to 0} \cos^{-1}\left(\dfrac{\sin x}{x}\right) = \cos^{-1}\left(\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right) = \cos^{-1}(1) = \underline{\underline{0}}.$

Intermediate Value Theorem



If $f(x)$ is cts at all pts in $[a,b]$ ("cts on $[a,b]$") and $f(a) < c < f(b)$ then there exists

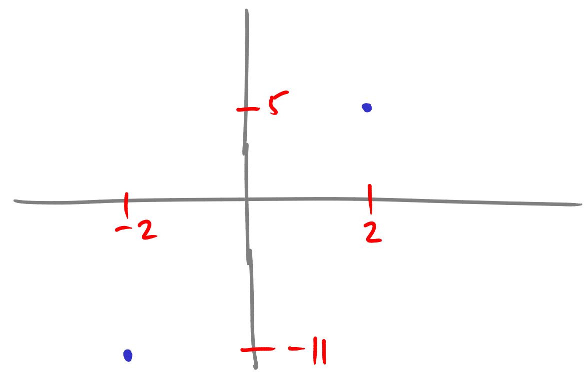
Some x in $[a,b]$ with $f(x) = c$.

<u>Ex</u> How do we solve $x^3 - x^2 + 1 = 0$?

Write $f(x) = x^3 - x^2 + 1$

$f(-2) = -8 - 4 + 1 = -11$

$f(2) = 8 - 4 + 1 = 5$



so IVT guarantees that

$x^3 - x^2 + 1 = 0$

has <u>at least</u> one solution x in $[-2, 2]$.

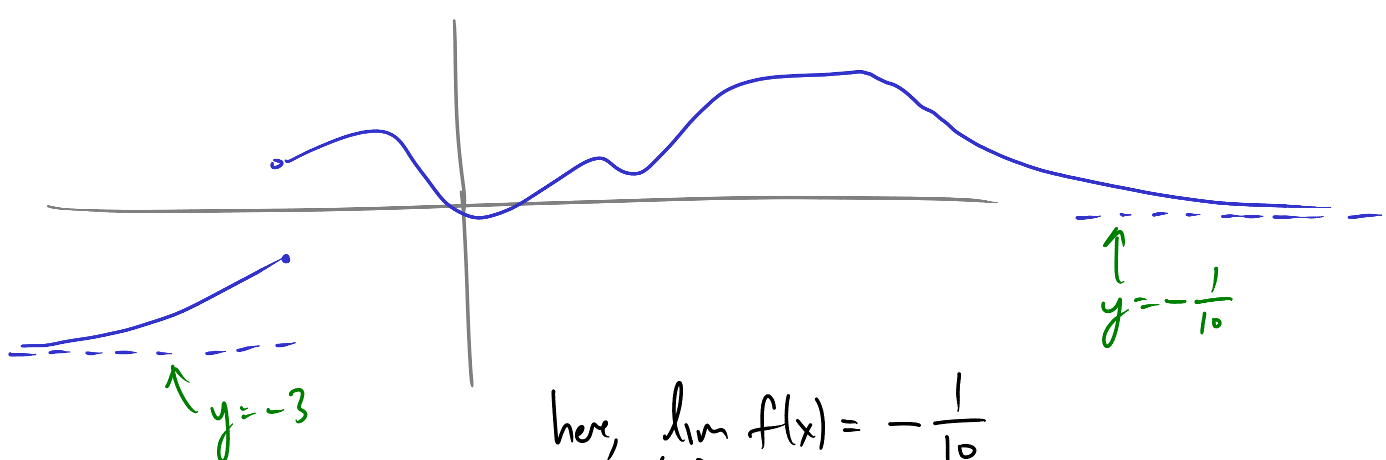
<u>Ex</u> $\lim_{x \to 2} 2^{\sqrt{x^2+12}} = 2^{\sqrt{4+12}} = 2^{\sqrt{16}} = 2^4 = \underline{16}$

(because $f(x) = 2^{\sqrt{x^2+12}}$ is cts everywhere on its domain, ρ^{the} at $a=2$)

<u>Limits as $x \to \pm\infty$</u>

We say $\lim_{x \to \infty} f(x) = L$ if, as x grows without bound in <u>positive</u> direction, $f(x)$ approaches L.

$\lim_{x \to -\infty} f(x) = L$ if, " " " " " " <u>negative</u> " " " ".



here, $\lim_{x \to \infty} f(x) = -\frac{1}{10}$

$\lim_{x \to -\infty} f(x) = -3$