Lecture 11

24 Sep 2018

Midterm 1 next Mon (1 week from today)

Covers material from Lectures 1–11 (thru trig derivatives)

To prepare. ① go over + redo HW's (exam has ~12 Quest-like problems)

② get extra probs from text sections we covered

HW05 will be due Tue night, not Mon night — contains material from Lecture 11 (examinable)

My office hr: today (2–3, RLM 9.134)

Last time: deriv. of polynomials, powers, exponentials, product rule

Ex $$\frac{d}{dx}\underset{f}{x^3}\underset{g}{e^x} = \underset{f'}{3x^2}\underset{g}{e^x} + \underset{f}{x^3}\underset{g'}{e^x}$$

Quotient Rule: if f and g are differentiable at x and $g(x) \neq 0$

then $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\frac{d}{dx}\left(\frac{\text{high}}{\text{low}}\right) = \frac{\text{low}\frac{d}{dx}\text{high} - \text{high}\frac{d}{dx}\text{low}}{\text{low}^2}$$

Ex $$\frac{d}{dx}\left(\frac{\overset{f}{1}}{\underset{g}{x}}\right) = \frac{\overset{g}{x}\cdot\overset{f'}{0} - \overset{f}{1}\cdot\overset{g'}{1}}{\underset{g^2}{x^2}} = -\frac{1}{x^2}$$

(check: $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1\cdot x^{-2}$
$= -\frac{1}{x^2}$ ✓)

Ex $$\frac{d}{dx}\left(\frac{\overset{f}{x^2+3x+5}}{\underset{g}{1+x}}\right) =$$

$$= \frac{(1+x)(2x+3) - (x^2+3x+5)(1)}{(1+x)^2}$$

$$= \frac{(2x^2+5x+3) - (x^2+3x+5)}{(1+x)^2}$$

$$\frac{d}{dx}\left(\frac{x^2+3x+2}{1+x}\right) = ?$$

$$= \dots$$

$$= \frac{x^2+2x+1}{(1+x)^2} = \frac{(x+1)^2}{(x+1)^2} = \underline{\underline{1}}$$

(why?)

$$= \frac{x^2+2x-2}{(1+x)^2}$$

Ex $$\frac{d}{dx}\left(\frac{e^x}{x-1}\right) = \frac{(x-1)e^x - e^x(1)}{(x-1)^2} = \frac{xe^x - e^x - e^x}{(x-1)^2} = e^x\,\frac{x-2}{(x-1)^2}$$

Ex $$\frac{d}{dx}\left(\frac{e^x+e^x x^3}{\sqrt{x}}\right) = \frac{\sqrt{x}\,\frac{d}{dx}(e^x+e^x x^3) - (e^x+e^x x^3)\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}$$

$(x>0)$

$$= \frac{\sqrt{x}\,(e^x+e^x x^3+e^x 3x^2) - (e^x+e^x x^3)\frac{1}{2\sqrt{x}}}{x}$$

$$= \cdots$$

$$\left(\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\right)$$

OR, can just split it up!

from $\frac{x^3}{\sqrt{x}} = \frac{x^3}{x^{\frac{1}{2}}} = x^{3-\frac{1}{2}} = x^{\frac{5}{2}}$

$$\frac{d}{dx}\left(\frac{e^x+e^x x^3}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{e^x}{\sqrt{x}} + \frac{e^x x^3}{\sqrt{x}}\right) = \frac{d}{dx}\left(e^x x^{-1/2} + e^x x^{5/2}\right) = \cdots$$

just use product rule!

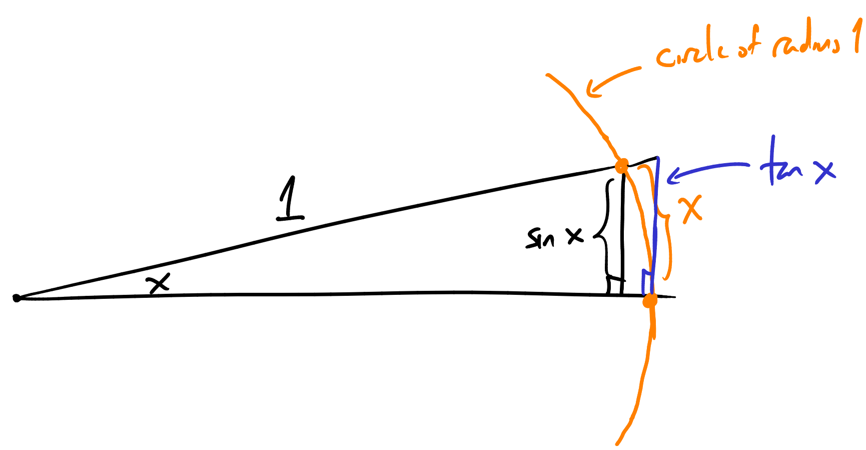
[nb $e^x x^{\frac{5}{2}} \neq (xe)^{x+\frac{5}{2}}$!]

Derivatives of Trig Functions

Need to work out a few limits 1st,

Fact $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

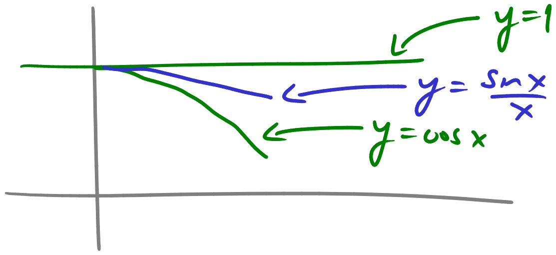
Why? $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$



From the picture: $\sin x < x < \tan x$

So $\sin x < x$ and $x < \tan x = \frac{\sin x}{\cos x}$

$\rightarrow \frac{\sin x}{x} < 1$ $\qquad$ $\cos x < \frac{\sin x}{x}$ $\qquad$ ie $\cos x < \frac{\sin x}{x} < 1$



So $\lim_{x\to 0} \cos x \le \lim_{x\to 0} \frac{\sin x}{x} \le \lim_{x\to 0} 1$

$$1 \le \lim_{x\to 0} \frac{\sin x}{x} \le 1$$

So $\underline{\lim_{x\to 0} \frac{\sin x}{x} = 1}$.

Fact $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$.

Why?
$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1}$$
$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$$
$$= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$
$$= \lim_{x\to 0} -\frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}$$
$$= (-1)\cdot\left(\frac{0}{2}\right) = \underline{\underline{0}}$$

$$\left(\begin{array}{l} \sin^2 x + \cos^2 x = 1 \\ \cos^2 x - 1 = -\sin^2 x \end{array}\right)$$

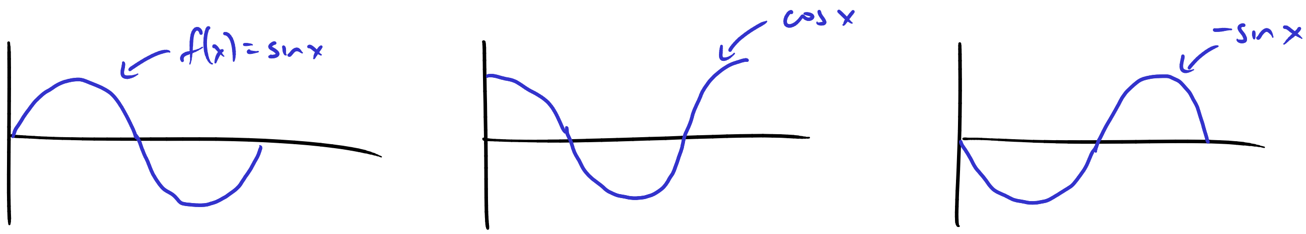
$$\left(\begin{array}{l} \sin^2 x = \sin x \cdot \sin x \\ \sin(x^2) = \sin(x \cdot x) \\ \qquad \ne \sin^2 x \end{array}\right)$$

Fact $\frac{d}{dx}(\sin x) = \cos x$.

Why?
$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \lim_{h\to 0} \sin x \cdot \left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \left(\frac{\sin h}{h}\right)$$
$$= \sin x \cdot 0 + \cos x \cdot 1$$
$$= \underline{\underline{\cos x}}$$

Ex $\frac{d}{dx}(x^4 \sin x) = 4x^3 \sin x + x^4 \cos x$. $\quad (= x^3(4\sin x + x\cos x))$

Fact $\frac{d}{dx}(\cos x) = -\sin x.$ (similar derivation)



All other deriv. of trig functions follow from these.

Ex $$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Summary

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$