Lecture 12

26 Sep 2018

Exam 1 next Monday in class (here)

Practice exam pbs will be posted on Quest today

No calculators on the exam!

My office hrs: today 3:30–4:30

Friday 4–5

EXTRA

Last time: derivatives of trig functions $\frac{d}{dx}(\sin x) = \cos x$ ...

Ex Where does the graph $y = \frac{\sec x}{1+\tan x}$ have a horizontal tangent?

Need to solve $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = \frac{(1+\tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

$\sin^2 x + \cos^2 x = 1$

$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$\tan^2 x + 1 = \sec^2 x$

$\sec^2 x - \tan^2 x = 1$

so $\tan^2 x - \sec^2 x = -1$

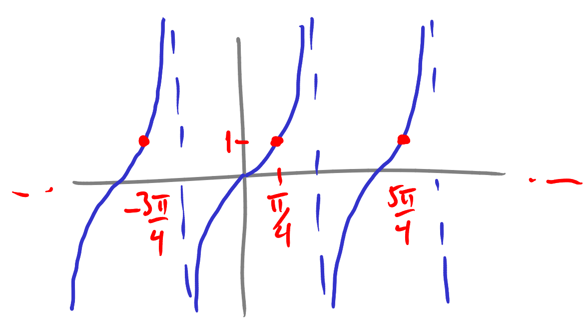
when is this $= 0$? When $\tan x = 1$ $\left(\sec x = \frac{1}{\cos x} \text{ never } 0\right)$

eg at $x = \pi/4$ since $\tan\left(\frac{\pi}{4}\right) = 1$

and $\tan x$ is periodic with period π so $\tan\left(\frac{\pi}{4} + n\pi\right) = 1$ for any n

So, get horizontal tangents at $x = \frac{\pi}{4} + n\pi$ for any n

(and no others)



Q What is the 21st derivative of $f(x) = \sin x + x^{15}$?

$$f'(x) = \cos x + 15x^{14}$$

$$f''(x) = -\sin x + 15\cdot 14 x^{13}$$

$$f^{(3)}(x) = -\cos x + 15\cdot 14\cdot 13 x^{12}$$

$$f^{(4)}(x) = \sin x + 15\cdot 14\cdot 13\cdot 12 x^{11}$$

$$\vdots$$

$$f^{(8)}(x) = \sin x + (- \quad \text{—} \quad) x^7$$

$$\vdots$$

$$f^{(20)}(x) = \sin x$$

$$f^{(21)}(x) = \underline{\cos x}$$

Remark:
This "self-reproducing behavior" is a hint that trig functions and exponentials are secretly closely connected. Indeed, $e^{ix} = \cos x + i \sin x$

("Euler's formula")

Recall $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

What is $\lim_{x\to 0} \frac{\sin 3x}{3x}$?

Let $u = 3x$. Then $\lim_{x\to 0} \frac{\sin 3x}{3x} = \lim_{u\to 0} \frac{\sin u}{u} = \underline{1}$.

Ex $\lim_{x\to 0} \frac{\sin 3x}{x} = ?$

Let $u = 3x$ Then $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{u\to 0} \frac{\sin u}{\left(\frac{u}{3}\right)} = \lim_{u\to 0} 3\, \frac{\sin u}{u} = 3\cdot 1 = \underline{3}$

$\frac{u}{3} = x$

Ex $\lim_{x\to 0} \frac{\sin 7x}{2x} = \underline{\frac{7}{2}}$ (take $u = 7x$)

Chain Rule

How to get the derivative of $F(x) = \sqrt{1+x^2}$?

Think of $F(x)$ as $f \circ g(x)$ ie $f(g(x))$ where $g(x) = 1+x^2$, $f(u) = \sqrt{u}$

Chain Rule: if g is diff'ble at x
and f is diff'ble at $g(x)$
and $F = f \circ g$
then $F'(x) = f'(g(x))\, g'(x)$

Or: if $y = f(u)$, $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Ex $y = \sqrt{1+x^2}$ — what is $\dfrac{dy}{dx}$? Say $u = 1+x^2$, $y = \sqrt{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot 2x = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \underline{\underline{\frac{x}{\sqrt{1+x^2}}}}$$

Ex $y = \sin^2 x$ Say $u = \sin x$, $y = u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 2u \cdot \cos x = \underline{\underline{2 \sin x \cos x}} = \underline{\underline{\sin 2x}} \quad \text{(why?)}$$

(we could also do this by product rule.
$y = \sin x \cdot \sin x$ so $\dfrac{dy}{dx} = \cos x \cdot \sin x + \sin x \cdot \cos x = 2 \sin x \cos x$ ✓)

[Shorthand: $y = (\sin x)^2$

$$\frac{dy}{dx} = \underbrace{2 \sin x}_{\text{usual power rule formula}} \cdot \overbrace{\cos x}^{\text{derivative of "inside part" } \sin x}$$
]

Ex $y = e^{\sin x}$ $\qquad \dfrac{dy}{dx} = e^{\sin x} \cdot \cos x$

derivative of e^u is e^u

derivative of "inside part" $u = \sin x$, $\dfrac{du}{dx} = \cos x$

Ex $y = e^{\cos 2x}$

$$\frac{dy}{dx} = e^{\cos 2x} \cdot \frac{d}{dx}(\cos 2x)$$
$$= e^{\cos 2x} \cdot (-\sin 2x) \cdot 2$$
$$= \underline{\underline{-2\sin 2x\, e^{\cos 2x}}}$$

$$\left[\; \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \qquad \left.\begin{aligned} y &= e^u \\ u &= \cos v \\ v &= 2x \end{aligned}\right\} \to \begin{aligned} \frac{dy}{dx} &= e^u(-\sin v) \cdot 2 \\ &= e^{\cos 2x}(-\sin 2x) \cdot 2 \end{aligned} \;\right]$$

Ex $y = \sin(x^2)$

$$\frac{dy}{dx} = \cos(x^2) \cdot 2x$$

$\frac{d}{du} \sin u = \cos u$

deriv. of "inside" x^2

Ex $y = (x^2 - 3)^{170}$

$$\frac{dy}{dx} = 170(x^2-3)^{169} \cdot 2x$$

$$\frac{d^2y}{dx^2} = 170\left(169(x^2-3)^{168} \cdot 2x \cdot 2x + 170(x^2-3)^{169} \cdot 2\right)$$

$$= \cdots$$

$$= 340(x^2-3)^{168}(339x^2-3)$$