Lecture 17

10 Oct 2018

(A) $\frac{d}{dx} \ln\left(\frac{(x-8)^{13}}{(x^2+2)^6}\right)$

Hard way: chain rule — $\frac{1}{\frac{(x-8)^{13}}{(x^2+2)^6}} \cdot \frac{d}{dx}\left(\frac{(x-8)^{13}}{(x^2+2)^6}\right)$ + quotient rule...

Easy way: $\frac{d}{dx}\left(\ln (x-8)^{13}\right) - \frac{d}{dx}\left(\ln (x^2+2)^6\right)$

Even Easier way: $13 \frac{d}{dx} \ln(x-8) - 6 \frac{d}{dx} \ln(x^2+2)$

$$= \frac{13}{x-8} - 6 \cdot \frac{1}{x^2+2} \cdot 2x$$

$$= \frac{13}{x-8} - \frac{12x}{x^2+2}$$

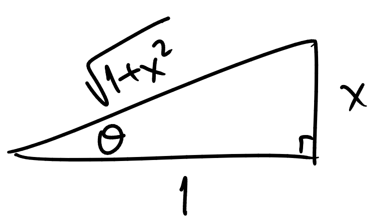
(B) $\frac{d}{dx}\left(\frac{x}{13}\right) = \frac{1}{13}$ (much easier than doing quotient rule!)

(C) How to simplify $\underbrace{\sin(\tan^{-1} x)}$? ie if $\theta = \tan^{-1} x$ what is $\sin\theta$?

$\uparrow$ θ

ie if $\tan\theta = x$ what is $\sin\theta$?

$\tan\theta = \tan(\tan^{-1} x) = x$



$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{x}{1}$$

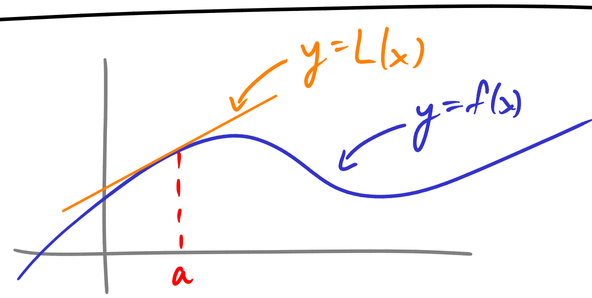
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{1+x^2}}$$

So, $\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$

Last time: linear approximation

if we know $f(a)$ and $f'(a)$

then let $L(x) = f(a) + f'(a)(x-a)$



$L(x)$ is a linear approximation to $f(x)$.

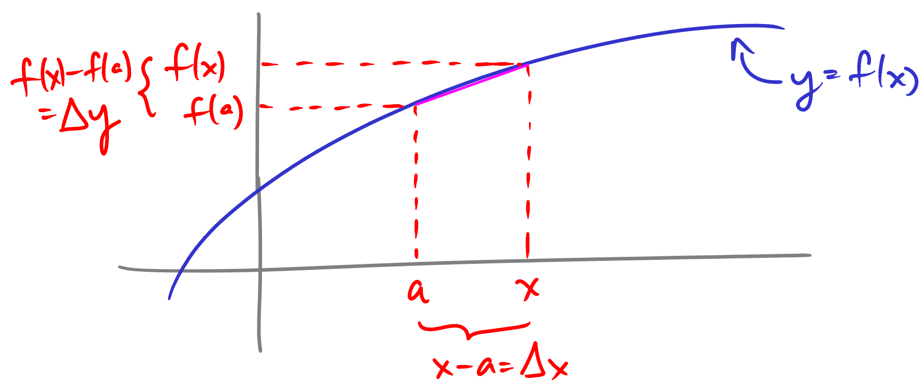
Ex Estimate $\sqrt[3]{28}$. We'll do this by linear approx of $f(x)=\sqrt[3]{x}$, at the point $a=27$.

$$L(x) = f(27) + f'(27)(x-27)$$
$$= 3 + \frac{1}{27}(x-27)$$

$$f(27) = \sqrt[3]{27} = 3$$
$$f'(x) = \frac{1}{3}x^{-2/3}$$
$$f'(27) = \frac{1}{3}\left(27^{-2/3}\right)$$
$$= \frac{1}{3}\left(27^{\frac{1}{3}}\right)^{-2}$$
$$= \frac{1}{3}\left(3^{-2}\right)$$
$$= \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27}$$

So $\sqrt[3]{28} \approx L(28) = 3 + \frac{1}{27}(28-27)$
$$= \underline{\underline{3 + \frac{1}{27}}}$$

Another way to think about this:



$$f(x) = f(a) + \Delta y$$
$$= f(a) + \left(\frac{\Delta y}{\Delta x}\right)\Delta x$$

If Δx is small then $\frac{\Delta y}{\Delta x}$ is about $\frac{dy}{dx}$ $\quad\left(\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\right)$

So $f(x) \approx f(a) + \left(\frac{dy}{dx}\right)\Delta x$
$$= f(a) + f'(a)\,\Delta x$$
$$= f(a) + f'(a)(x-a)$$

Differential of f: (just notation)

write "$df = \frac{df}{dx}\,dx$" as analog of $\quad \Delta f \approx \frac{df}{dx}\Delta x$ for small Δx

think of df as "infinitesimally small" Δf
dx as Δx

e.g. $d(\tan x) = \frac{d(\tan x)}{dx} \cdot dx = \sec^2 x \, dx$

$$d(\tan x) = \sec^2 x \, dx$$

so for small Δx, $\Delta(\tan x) \approx \sec^2 x \, \Delta x$

$\underline{\text{Ex}}$ Estimate $\ln(0.96)$ Idea: 0.96 is close to 1.

$\ln(1) = 0$.

$\Delta x = x - 1$

$= -0.04$.

$a = 1$

$x = 0.96$

$f'(a) = 1$

$x - a = -0.04$

$$d(\ln x) = \frac{d(\ln x)}{dx} \, dx = \frac{1}{x} \, dx.$$

$$\Delta(\ln x) \approx \frac{1}{x} \Delta x.$$

$$= \frac{1}{1} \cdot (-0.04)$$

$$= -0.04.$$

So $\ln(0.96) = \ln(x + \Delta x) \approx \ln(x) + \Delta(\ln x)$

$= \ln(1) + \Delta(\ln x)$

$= 0 - 0.04$

$= \underline{-0.04}.$

$\underline{\text{Ex}}$ What is the linear approx. to $f(x) = 3x^5$ at $a = 1$?

Need $f(1)$ and $f'(1)$.

$f(1) = 3$

$f'(x) = 15x^4$

$f'(1) = 15$

So

$$L(x) = f(1) + \overbrace{f'(1)(x-1)}^{\approx \Delta f}$$

$$= \underline{3 + 15(x-1)}$$

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so e.g. $f(1.01) \approx 3 + 15(1.01-1) = 3 + 15(.01) = \underline{\underline{3.15}}$

$\quad\quad\quad \overset{\parallel}{3(1.01)^5}$

To understand <u>why</u> it's true:

try <u>expanding out</u> $3(1.01)^5 = 3(1+.01)^5$

$$= 3\cdot(1+.01)(1+.01)(1+.01)(1+.01)(1+.01)$$

$$= 3\cdot\left(1 + 5\times .01 + (\text{much smaller terms})\right)$$

$$= 3\cdot(1.05 + \text{much smaller}) = \underline{\underline{3.15 + (\text{much smaller})}}$$

"$(dx)^2 = 0$"