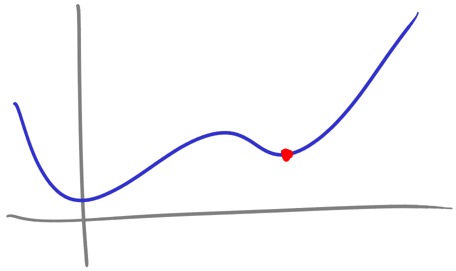
Lecture 20

17 Oct 2018

Last time: max and min



If $f(x)$ has a local min. at $x=a$

and $f(x)$ is diff'ble at $x=a$

and $x=a$ is not an endpoint of the domain of f

then $f'(a)=0$ (horiz tangent)

Strategy for finding absolute max/min for a function f with domain $[a,b]$:

1. find values of f at all "critical numbers": x where $f'(x)=0$ **or** $f'(x)$ DNE.
2. find values of $f(a)$, $f(b)$
3. take max, min values of f from that list.

Ex Find absolute max, min of

$f(x)=12+4x-x^2$ on $[0,5]$. $(0\leq x\leq 5)$

① $f'(x)$ exists everywhere — no points where $f'(x)$ DNE.

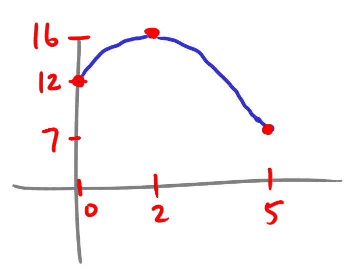
$f'(x)=4-2x$

so $f'(x)=0$ only at $4-2x=0$
$x=2$. So only critical # is $x=2$.

$f(2)=12+4\cdot 2-2^2=16.$

② $f(0)=12+4\cdot 0-0^2=12.$

$f(5)=12+4\cdot 5-5^2=7.$



③ max is $f(2)=16$. min is $f(5)=7$.

Ex Find absolute max of

$f(x) = x^{-2} \ln x$ for $1 \le x \le 100$.

① $f'(x) = -2x^{-3}\ln x + x^{-2}\cdot\frac{1}{x}$ exists everywhere on domain

$= -2x^{-3}\ln x + x^{-3}$

$= x^{-3}(-2\ln x + 1)$

$f'(x)=0$:

$0 = x^{-3}(-2\ln x+1)$ only if $0 = -2\ln x + 1$

$2\ln x = 1$

$\ln x = \frac{1}{2}$

$x = e^{\frac{1}{2}} = \sqrt{e}$

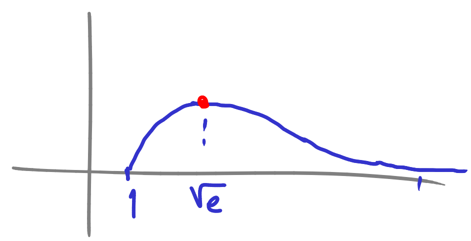
$f(\sqrt{e}) = (\sqrt{e})^{-2}\ln(\sqrt{e}) = \frac{1}{e}\cdot\frac{1}{2} = \frac{1}{2e}$

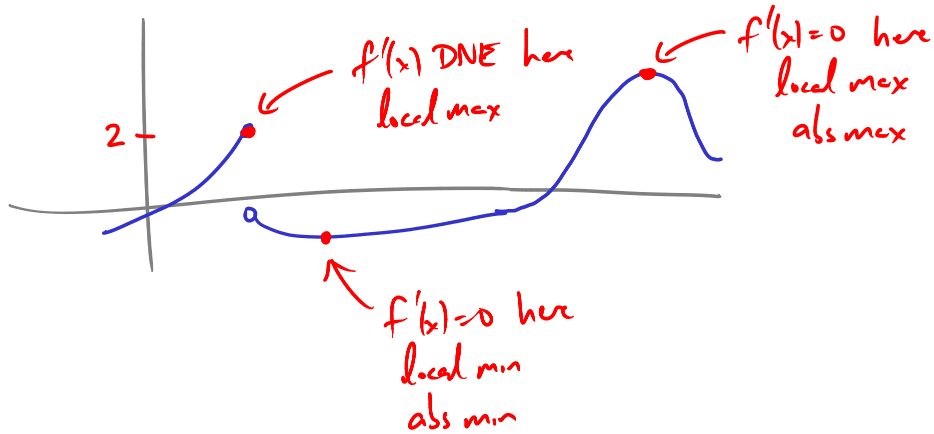
② $f(1) = 1\cdot\ln(1) = 0$

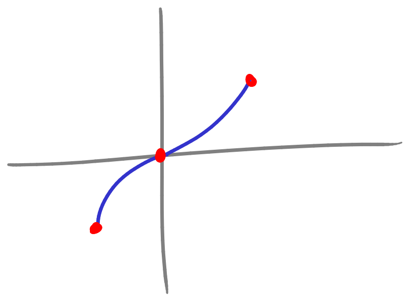
$f(100) = \frac{1}{10000}\ln 100$

③ The max is one of $\frac{1}{2e}, 0, \frac{\ln(100)}{10000}$. $\frac{1}{2e} > \frac{\ln(100)}{10000}$ so it's the max.

$f(\sqrt{e}) = \frac{1}{2e}$ is abs max.







$f(x) = x^3$ on $[-1,1]$:

$f'(x) = 3x^2$

$f'(x) = 0$ at $x = 0$

$f(0) = 0$

endpoints: $f(1) = 1$ ← abs max

$f(-1) = -1$ ← abs min

Graphing using derivatives

How do we use $f'(x)$ to get information about the graph of $f(x)$?

Ex Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x-2)(x+1)$$

To see if $f'(x)$ is +ve or −ve, look at

sign of $f'(x)$	$---$ $= -$		$--+$ $= +$		$+-+$ $= -$		$+++$ $= +$
x		-1		0		2	

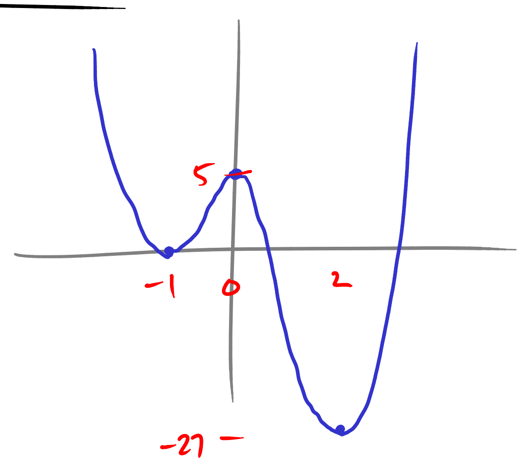
So $f(x)$ is increasing for $x \in (-1,0) \cup (2,\infty)$

decreasing for $x \in (-\infty,-1) \cup (0,2)$

$f(-1) = 0$

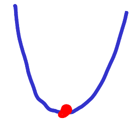
$f(0) = 5$

$f(2) = -27$



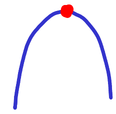
Let's look closer at the critical pts. $f'(x) = 0$ at $x = -1, 0, 2$.

At $x=2$: sign of $f'(x)$: $-$ | $+$; x: 2



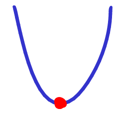
local min

At $x=0$: $+$ | $-$; 0



local max

At $x=-1$: $-$ | $+$; -1



local min

First Derivative Test

If c is a critical number for $f(x)$,

① If $f'(x)$ changes sign from $+$ to $-$ at c, then f has <u>local max</u> at c.

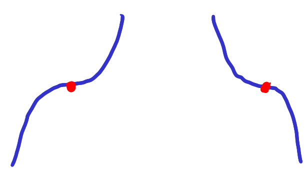
$+$ | $-$

② If $f'(x)$ changes sign from $-$ to $+$ at c, then f has <u>local min</u> at c.

$-$ | $+$

③ If $f'(x)$ <u>doesn't</u> change sign at c then f has <u>neither</u> <u>loc max</u> or <u>loc min</u> at c.

$+$ | $+$ $\qquad$ $-$ | $-$



<u>Ex</u> Find all local max/min of $f(x) = x^{\frac{1}{3}}(x+4)$ on $(0,\infty)$ <u>and</u> $(-\infty,0)$

For $x \neq 0$ $f'(x)$ exists everywhere.

$$f'(x) = \frac{1}{3}x^{-2/3}(x+4) + x^{1/3}$$

$$= \frac{1}{3}x^{1/3} + \frac{4}{3}x^{-2/3} + x^{1/3}$$

$$= \frac{4}{3}\left(x^{\frac{1}{3}} + x^{-\frac{2}{3}}\right) = \frac{4}{3}x^{-2/3}(x+1)$$

$\uparrow$ $= (x^{-1/3})^2 > 0$

$f'(x) = 0$ only at $x=-1$.

$\longrightarrow$ on $(0,\infty)$: <u>no</u> local max/min on $(-\infty, 0)$:

Sign of $f'(x)$	$+- = -$		$++ = +$	
x		-1		0

So $x=-1$ is local minimum.