Lecture 23 | 24 Oct 2018

Midterm exam 2: Nov 2 (next Friday)

Details soon

If you have notes from Mon you're willing to share, tell me!

Last time: curve sketching, L'Hospital's Rule

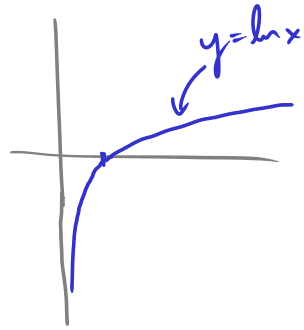
L'H Rule: If $\lim_{x\to a} f(x) = 0$, and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

[Why? If f, g are both diff'ble at a, then can see it:

Linear approximation: $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a) + (x-a)f'(a)}{g(a) + (x-a)g'(a)}$$
$$= \lim_{x\to a} \frac{0 + (x-a)f'(a)}{0 + (x-a)g'(a)}$$
$$= \lim_{x\to a} \frac{f'(a)}{g'(a)}$$]

Similarly if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

Ex $\lim_{x\to 0^+} \underset{\uparrow\, 0}{x}\ \underset{\uparrow\, -\infty}{\ln x} = ?$



$\frac{d}{dx}(x \ln x) = 1 + \ln x$

"$0 \cdot -\infty$" is a kind of indeterminate form — no help for calculating the limit

Trick: rewrite $x \ln x = \frac{\ln x}{\left(\frac{1}{x}\right)}$. Taking lim as $x \to 0^+$ now gives $\frac{-\infty}{\infty}$

[could also use $\frac{x}{\left(\frac{1}{\ln x}\right)}$]

So can use L'H rule:
$$\lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \left(\frac{-x^2}{x}\right)$$
$$= \lim_{x\to 0^+} (-x)$$

$= \underline{0}$.

So, $\lim_{x\to 0^+} x \ln x = 0$.

("as $x \to 0$, x goes to 0 <u>faster</u> than $\ln x$ goes to $-\infty$", if $x = \frac{1}{1000}$, $\underbrace{\underset{\frac{1}{1000}}{x} \cdot \underset{\approx -5}{\ln x}}$)

$\left[\tan x = \frac{\sin x}{\cos x} \approx \frac{x}{1} \text{ for small } x \right]$

<u>Ex</u> $\lim_{x\to 0} \frac{\tan x - x}{x^3} = ?$

Plug in $x=0$: $\frac{0-0}{0} = \frac{0}{0} \longrightarrow$ L'H Rule applies.

So, $= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$. Plug in $x=0$: $\frac{1^2-1}{0} = \frac{0}{0} \longrightarrow$ Could use L'H again.

Or, use trig identity: $\sec^2 x - 1 = \tan^2 x$

So,

$$= \lim_{x\to 0} \frac{\tan^2 x}{3x^2} = \frac{1}{3} \lim_{x\to 0} \frac{\tan^2 x}{x^2} = \frac{1}{3} \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{\cos^2 x}$$

$1^2 = 1$ $\quad$ $\frac{1}{1} = 1$

$\left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$

$$= \frac{1}{3}.$$

So, $\lim_{x\to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}$.

(Could use this to get a better estimate of $\tan x$ for x near 0:

$\tan x - x \approx \frac{1}{3} x^3$

ie $\tan x \approx x + \frac{1}{3} x^3$

so, $\tan(0.1) \approx 0.1 + \frac{1}{3}(0.001)$

$= 0.10033.$)

<u>Indeterminate Powers</u>

$\lim_{x\to a} f(x)^{g(x)} = ?$ $\quad$ eg $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = ?$

Sometimes can do this by "plugging in", sometimes not.

We may get indeterminate forms like ∞^0, 0^0, 1^∞.

A trick: try taking log.

(and $L \neq 0$)

ie, if $L = \lim_{x \to a} f(x)^{g(x)}$ then $\ln L = \ln\left(\lim_{x\to a} f(x)^{g(x)}\right)$

$$= \lim_{x \to a} \ln\left(f(x)^{g(x)}\right)$$

$$= \lim_{x \to a} \left(g(x) \ln f(x)\right)$$

which might be easier to calculate.

Ex $\lim_{x \to 0^+} x^x = ?$ "0^0"

let $L = \lim_{x \to 0^+} x^x$

then $\ln L = \ln\left(\lim_{x \to 0} x^x\right) = \lim_{x \to 0} \ln(x^x) = \lim_{x \to 0} x \ln x = 0$ (previous example)

So $\ln L = 0$

ie $L = e^0 = 1$. So $\lim_{x \to 0^+} x^x = 1$.

Ex $\lim_{x \to 0^+} (1+\sin 4x)^{\cot x} = ?$ plugging gives "1^∞" — indeterminate

$L = \lim_{x \to 0^+} (1+\sin 4x)^{\cot x}$

$\ln L = \lim_{x \to 0^+} \ln\left((1+\sin 4x)^{\cot x}\right)$

$$= \lim_{x \to 0^+} \underbrace{\cot x}_{\infty} \cdot \ln \underbrace{(1+\sin 4x)}_{0}$$

$$= \lim_{x \to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \quad \frac{0}{0} \text{ use L'H:}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1+\sin 4x} \cdot 4\cos 4x}{\sec^2 x} = \frac{\frac{1}{1+0} \cdot 4 \cdot 1}{1^2} = 4.$$

So $\ln L = 4$.

$L = e^4$.

1e $\lim_{x \to 0^+} (1+\sin 4x)^{\cot x} = \underline{\underline{e^4}}.$