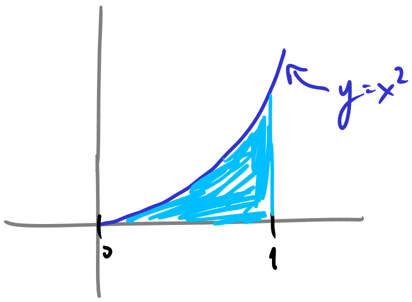
Lecture 28 — 7 Nov 2018

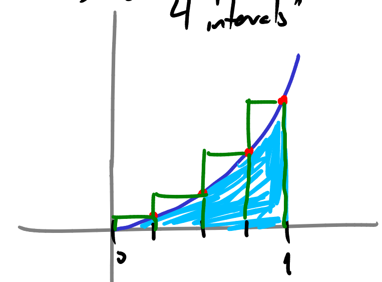
Last time: estimating area between graph of $y=f(x)$ and the x-axis

ex area between $y=x^2$ and x-axis for $x \in [0,1]$

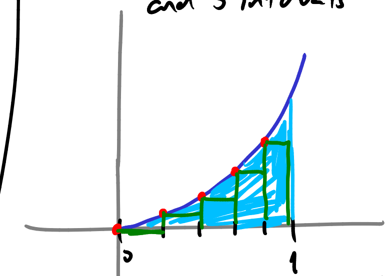


Approximate by a bunch of rectangles:

"using right endpoints and 4 intervals"



"using left endpoints and 5 intervals"



$$A = \frac{1}{4}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + (1)^2\right] = \frac{15}{32}$$

$R_4 = \frac{15}{32}$ overest

$$A = \frac{1}{5}\left[0^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2\right] = \frac{30}{125} = \frac{6}{25}$$

$L_5 = \frac{6}{25}$ underest

the actual area A has $\frac{6}{25} < A < \frac{15}{32}$

Now say we use 100 rectangles.

Then get
$$L_{100} = \underbrace{\frac{1}{100}}_{\text{width of each rect.}}\left[\underbrace{0^2 + \left(\frac{1}{100}\right)^2 + \left(\frac{2}{100}\right)^2 + \left(\frac{3}{100}\right)^2 + \cdots + \left(\frac{99}{100}\right)^2}_{\text{heights of rectangles}}\right] \approx 0.3285$$

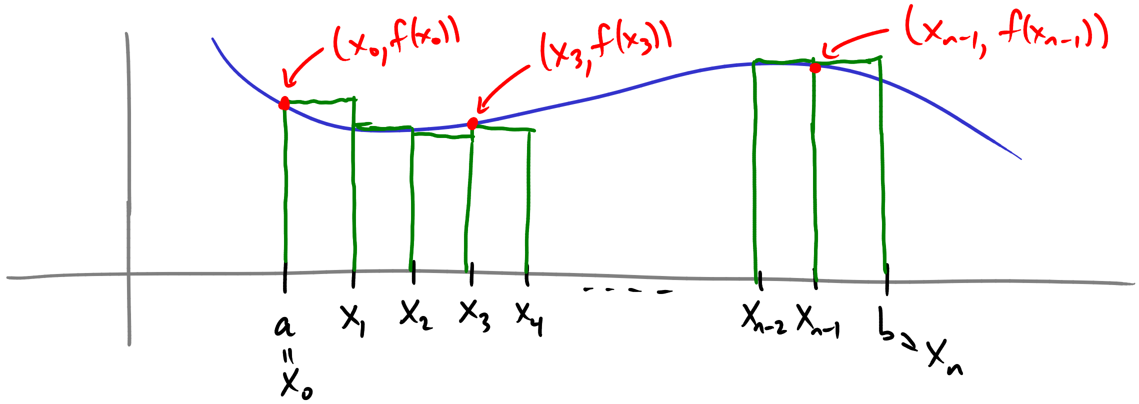
$$R_{100} = \frac{1}{100}\left[\left(\frac{1}{100}\right)^2 + \left(\frac{2}{100}\right)^2 + \left(\frac{3}{100}\right)^2 + \cdots + \left(\frac{99}{100}\right)^2 + 1^2\right] \approx 0.3385$$

n	L_n	R_n
10	.285	.385
100	.3285	.3385
1000	.33285	.33385
10000	.333285	.333385

As $n \to \infty$, both L_n and R_n approach $\frac{1}{3}$.

So, $\frac{1}{3}$ is the exact area between $y=x^2$ and the x-axis between $x=0$, $x=1$.

For any continuous function $f(x)$, estimate the area similarly: chop the interval $[a,b]$ into n subintervals



Width of each rect: $\Delta x = \frac{b-a}{n}$

Heights of rects: (using left endpoints) $f(x_0), f(x_1), f(x_2), \ldots, f(x_{n-1})$

where

$x_0 = a$
$x_1 = a + \Delta x$
$x_2 = a + 2\Delta x$
$\vdots$

$x_i = a + i \cdot \Delta x = a + i \cdot \frac{b-a}{n}$

$\longrightarrow$ estimated area. $L_n = \Delta x \cdot \left(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-2}) + f(x_{n-1}) \right)$

A convenient notation ("sigma notation"):

the symbol $\displaystyle\sum_{i=1}^{n} a_i$ means $a_1 + a_2 + a_3 + \cdots + a_n$.

Ex What is $\displaystyle\sum_{i=1}^{4} i^2$?

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = \underline{\underline{30}}.$$

Ex What is $\displaystyle\sum_{i=1}^{6} 3i$?

$3(1) + 3(2) + 3(3) + 3(4) + 3(5) + 3(6)$
$= \underline{\underline{63}}.$

Ex Write $\frac{1^3}{n} + \frac{2^3}{n} + \frac{3^3}{n} + \cdots + \frac{n^3}{n}$ in sigma notation.

$$\sum_{i=1}^{n} \frac{i^3}{n}$$

$f(x_0)+f(x_1)+\cdots+f(x_{n-1})$

In this notation, $L_n = \Delta x \sum_{i=1}^{n} f(x_{i-1})$.

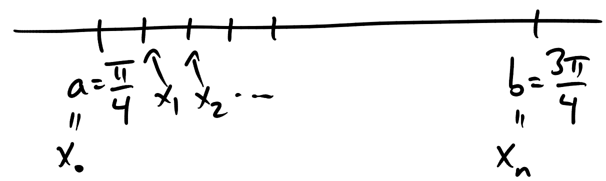
$R_n = \Delta x \sum_{i=1}^{n} f(x_i)$.

$f(x_1)+f(x_2)+\cdots+f(x_n)$.

The actual area is $A = \lim_{n\to\infty} L_n$ or, $\lim_{n\to\infty} R_n$. (Both are the same!)

Ex Let A be the area of the region under the graph of $f(x)=\sin^2 x$ between $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$. Using right endpoints as sample points,

- Write a formula for A as a limit.



$a=\frac{\pi}{4}$ $\quad b=\frac{3\pi}{4}$

$$\Delta x = \frac{b-a}{n} = \frac{\frac{3\pi}{4}-\frac{\pi}{4}}{n} = \frac{\pi}{2n}$$

$$x_i = a + i\Delta x = \frac{\pi}{4} + i\frac{\pi}{2n}$$

$$R_n = \Delta x \cdot \sum_{i=1}^{n} f(x_i) = \frac{\pi}{2n}\cdot\sum_{i=1}^{n}\sin^2\left(\frac{\pi}{4}+i\frac{\pi}{2n}\right)$$

The actual area is $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{\pi}{2n}\sum_{i=1}^{n}\sin^2\left(\frac{\pi}{4}+i\frac{\pi}{2n}\right)$

- Estimate A using 3 rectangles.

$$R_3 = \frac{\pi}{6}\cdot\sum_{i=1}^{3}\sin^2\left(\frac{\pi}{4}+i\frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}\left(\sin^2\left(\frac{\pi}{4}+\frac{\pi}{6}\right)+\sin^2\left(\frac{\pi}{4}+2\frac{\pi}{6}\right)+\sin^2\left(\frac{\pi}{4}+3\frac{\pi}{6}\right)\right)$$

$$\approx 1.23885.$$

Definite integrals

Say $f(x)$ is a function defined for $a \le x \le b$.

Divide $[a,b]$ into n equal subintervals of width Δx endpoints $x_0, x_1, x_2, \ldots, x_n$

$$\Delta x = \left(\frac{b-a}{n}\right) \qquad (x_i = x_0 + i\Delta x)$$

Pick any "sample points" x_i^* in $[x_{i-1}, x_i]$.

The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Bigg(\underbrace{\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x}_{\text{"Riemann sum"}} \Bigg)$$

if that limit exists!

(It always exists, if f is continuous.)