Lecture 37 — 3 Dec 2018

Midterm 3 Fri 7 Dec
covers Lectures 24–36

Last time: computing volumes by integration

Average values

What do we mean by the average value of some function f?

e.g. "average temperature over a day" —— $f(t)$ = temperature (in °F) at time t (in s)

What do we do to f to get the average?

Average of a finite collection of numbers:

average of $\{2,4\}$ is $\frac{2+4}{2} = \frac{6}{2} = 3$

" " $\{2,4,7\}$ is $\frac{2+4+7}{3} = \frac{13}{3}$

" " $\{y_1, y_2, \ldots, y_n\}$ is $\frac{y_1+y_2+\cdots+y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} y_i$

To define average of a function $f(x)$ on the domain $[a,b]$: divide interval into n parts

take the average of the sample values: $y_i = f(x_i^*)$

so the approximate average of f is $\frac{y_1+y_2+\cdots+y_n}{n} = \frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{n}$

$$= \sum_{i=1}^{n} f(x_i^*)\cdot\left(\frac{1}{n}\right)$$

Looks like a Rieman sum!

If we wanted to evaluate $\int_a^b f(x)\,dx$ we would write Riem sums $\sum_{i=1}^{n} f(x_i^*)\cdot \Delta x$

$$= \sum_{i=1}^{n} f(x_i^*)\cdot\left(\frac{b-a}{n}\right)$$

Comparing the two: $\int_a^b f(x)\,dx = (b-a)\cdot$ (the average value of $f(x)$ on interval $[a,b]$)

ie:

The average value of $f(x)$ on the interval $[a,b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

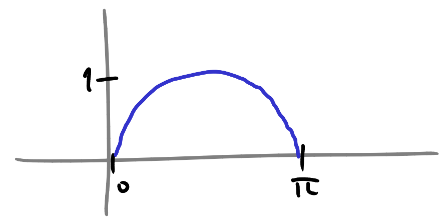
Ex The average value of $f(x) = \sin x$ on $[0, \pi]$ is

N

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin x \, dx$$

$$= \frac{1}{\pi}\left(-\cos x \Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(-(-1) - (-1)\right) = \frac{1}{\pi}(2) = \underline{\underline{\frac{2}{\pi}}}$$



Ex The average value of $f(x) = c$ (c constant) over $[a,b]$ is

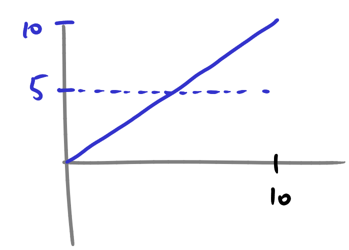
$$\frac{1}{b-a}\int_a^b c \, dx = \frac{1}{b-a}\cdot\left(cx \Big|_a^b\right)$$

$$= \frac{1}{b-a}\left(cb - ca\right)$$

$$= \frac{1}{b-a}\cdot c(b-a) = \underline{\underline{c}}$$

Ex The avg value of $f(x) = x$ over $[0, 10]$ is...

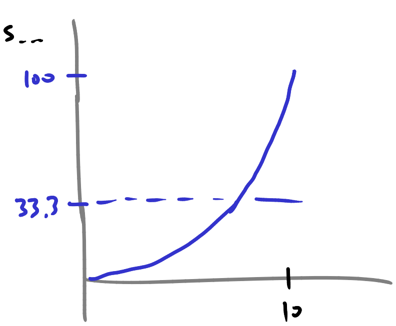
$$\frac{1}{10-0}\int_0^{10} x \, dx = \frac{1}{10}\left(\frac{x^2}{2}\Big|_0^{10}\right) = \frac{1}{10}\left(\frac{100}{2}\right) = \frac{1}{10}(50) = \underline{\underline{5}}$$



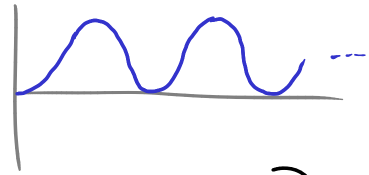
Ex The avg value of $f(x) = x^2$ over $[0, 10]$ is...

$$\frac{1}{10-0}\int_0^{10} x^2 \, dx$$

$$= \frac{1}{10}\left(\frac{x^3}{3}\Big|_0^{10}\right) = \frac{100}{3} \approx 33.3$$



Ex What is the average value of $f(x) = \sin^2 x$ over $[0, 2\pi]$?



First method:

$$\frac{1}{2\pi - 0}\int_0^{2\pi} \sin^2 x \, dx$$

$$\left[\begin{array}{l} \text{try } u = \sin x \\ du = \cos x \, dx \\ dx = \frac{du}{\cos x} \end{array} \longrightarrow \frac{1}{2\pi}\int u^2 \frac{du}{\cos x} = \frac{1}{2\pi}\int u^2 \frac{du}{\sqrt{1-u^2}} \longrightarrow \underline{\text{O HELP}}\right]$$

Use trig identity:

$$\cos 2x = -2\sin^2 x + 1$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

So have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{4\pi}\int_0^{2\pi} 1 - \cos 2x \, dx$$

$$= \frac{1}{4\pi}\left(x - \frac{1}{2}\sin 2x \Big|_0^{2\pi}\right)$$

$$= \frac{1}{4\pi}\Big((2\pi - 0) - (0 - 0)\Big) = \underline{\underline{\frac{1}{2}}}.$$

Second method:

$\sin^2 x + \cos^2 x = 1$

So their averages should also add up to 1

and their averages should be the same

so both must be $\underline{\underline{\frac{1}{2}}}$.