Sometimes we have to use both subst. <u>and</u> $\int$ by parts.

<u>Ex</u> $\int \cos(\sqrt{x})\,dx = ?$

Try substitution $u = \sqrt{x} = x^{\frac{1}{2}}$.

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad \text{so} \qquad dx = 2\sqrt{x}\,du$$

$$\int \cos(\sqrt{x})\,dx = \int \cos(u)\cdot 2\sqrt{x}\,du$$

$$= \int \cos(u)\cdot 2u\,du$$

Just to avoid confusion, let's change the name of the variable $u \to t$

So now $t = \sqrt{x}$

$$\int \cos(t)\cdot 2t\,dt$$

Int. by parts:

$$u = 2t \qquad v = \sin(t)$$
$$du = 2dt \qquad dv = \cos(t)\,dt$$

Then $\int \cos(t)\cdot 2t = \int u\,dv = uv - \int v\,du$

$$= 2t\sin(t) - \int 2\sin(t)\,dt$$

$$= 2t\sin(t) + 2\cos(t) + C$$

Remember $t = \sqrt{x}$:

$$= \underline{\underline{2\sqrt{x}\,\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C}}$$

Ex $\displaystyle\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta$

Try $t = \theta^2$

$dt = 2\theta\, d\theta \qquad\qquad d\theta = \frac{dt}{2\theta}$

$$= \int_{\frac{\pi}{2}}^{\pi} \theta^3 \cos(t)\, \frac{dt}{2\theta}$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \theta^2 \cos(t)\, dt$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} t\cos(t)\, dt \quad = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} u\, dv$$

$u = t \qquad v = \sin(t)$

$du = dt \qquad dv = \cos(t)\, dt$

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} u\, dv = \frac{1}{2}\, uv\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} v\, du$$

$$= \frac{1}{2}\left(t\sin(t)\right)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \sin(t)\, dt$$

$$= \frac{1}{2}\left(t\sin(t)\right)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2}\left(-\cos(t)\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{2}\left[t\sin(t) + \cos(t)\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{2}\left[\left[\pi(0) + -1\right] - \left[\frac{\pi}{2}(1) + 0\right]\right]$$

$$= \underline{-\frac{1}{2} - \frac{\pi}{4}}$$

Ex

$$\int \tan^{-1}(4t)\,dt = \int u\,dv$$

$$u = \tan^{-1}(4t) \qquad v = t$$

$$du = \frac{1}{1+(4t)^2}\cdot 4\,dt \qquad dv = dt$$

$$\int u\,dv = uv - \int v\,du$$

$$= \tan^{-1}(4t)\cdot t - \int t\cdot \frac{4}{1+(4t)^2}\,dt$$

$$= \quad '' \quad - \int \frac{4t}{1+(4t)^2}\,dt$$

$u = 4t$
$du = 4\,dt$
$dt = \frac{du}{4}$

$$= \tan^{-1}(4t)\cdot t - \int \frac{u}{1+u^2}\,\frac{du}{4}$$

$v = 1+u^2$
$dv = 2u\,du$
$du = \frac{dv}{2u}$

$$= t\cdot \tan^{-1}(4t) - \int \frac{u}{v}\,\frac{dv}{2u}\cdot\frac{1}{4}$$

$$= t\tan^{-1}(4t) - \frac{1}{8}\int \frac{dv}{v}$$

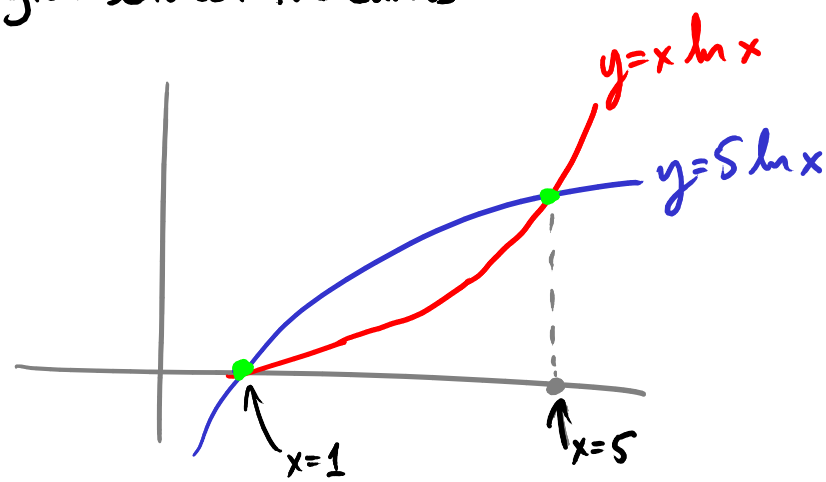
$$= t\tan^{-1}(4t) - \frac{1}{8}\ln|v|$$

$|v| = |1+u^2| = |1+16t^2|$

$$= t\tan^{-1}(4t) - \frac{1}{8}\ln(1+16t^2)$$

Ex Find the area of the region between the curves

$$y = 5\ln x$$
$$y = x\ln x$$



Intersection:

$5\ln x = x\ln x$

$x = 5$ or $\ln x = 0$

$x = 1$

$\ln(1) = 0$

$$\int_1^5 (5\ln x - x\ln x)\,dx$$

$$= \int_1^5 (5-x)\ln x\,dx$$

Int by parts: $u = \ln x$ $\quad v = 5x - \frac{1}{2}x^2$

$du = \frac{1}{x}\,dx$ $\quad dv = (5-x)\,dx$

$$\int_1^5 u\,dv = uv\Big|_1^5 - \int_1^5 v\,du$$

$$= (\ln x)\left(5x - \tfrac{1}{2}x^2\right)\Big|_1^5 - \underbrace{\int_1^5 \left(5x - \tfrac{1}{2}x^2\right)\frac{1}{x}\,dx}$$

$$= -\int_1^5 \left(5 - \tfrac{1}{2}x\right)dx$$

...

$$= \underline{\underline{\frac{25}{2}\ln 5 - 14}}$$

1 more substitution:

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$$

$u = 4+r^2$
$du = 2r\,dr$
$dr = \frac{du}{2r}$

$$= \int_4^5 \frac{r^3}{\sqrt{u}}\,\frac{du}{2r}$$

$$= \int_4^5 \frac{1}{2}\,\frac{r^2}{\sqrt{u}}\,du$$

$r^2 = u-4$

$$= \int_4^5 \frac{1}{2}\,\frac{u-4}{\sqrt{u}}\,du$$

$$= \int_4^5 \frac{1}{2}u^{1/2} - 2u^{-1/2}\,du$$

$$= \cdots$$

$$= -\frac{7}{3}\sqrt{5} + \frac{16}{3}$$