Lecture 31

12 Nov 2012

Housekeeping: Final exam dates posted online are now official (will send email soon)

Strategy tip: If you have a question

"Does the series $\sum_{n=1}^{\infty} a_n$
1) converge absolutely
2) converge conditionally
3) diverge"

usually first thing to do is check whether $\sum |a_n|$ converges! If it does the answer is 1).

Power Series (Sec 11.8)

A power series centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

"constants" "variable" constant

Ex $$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

is a power series centered at $a = 0$. (with $c_n = \frac{1}{n}$)

Ex $\sum_{n=0}^{\infty} n!\,(x-4)^n = (x-4)^0 + (x-4)^1 + 2\cdot(x-4)^2 + 6\cdot(x-4)^3 + \cdots$

$(0! = 1)$

is a power series centered at $a=4$.

Fact: For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are 3 possibilities:

1) Series converges only for $x=a$.
2) Series converges for *all* values of x.
3) There is some number R such that the series converges if $|x-a|<R$, diverges if $|x-a|>R$.

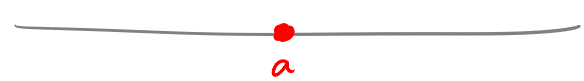
We call R the "radius of convergence" of the series.

In case 1, we say $R=0$.

In case 2, we say $R=\infty$.

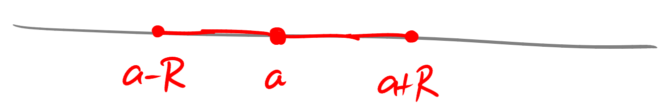
The "interval of convergence" is the set of all x where the series converges.

In case 1, it is just the point $x=a$.



In case 2, it is the whole # line, $-\infty<x<\infty$.
$x\in(-\infty,\infty)$.

In case 3, there are 4 possibilities for the interval of convergence:



$(a-R,\ a+R)$ $\qquad$ $(a-R,\ a+R]$

$[a-R,\ a+R)$ $\qquad$ $[a-R,\ a+R]$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = (x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} + \cdots$

Power series centered at $a=3$.

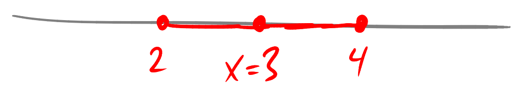
What is the interval of convergence?

Use Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|^{n+1}/n+1}{|x-3|^n/n} = |x-3| \cdot \frac{n}{n+1} \longrightarrow |x-3| \quad \text{as } n \to \infty$$

So, Ratio Test says series converges if $|x-3|<1$

diverges if $|x-3|>1$



i.e. the radius of convergence is $R=1$

Does the series converge at endpoints $x=2$ or $x=4$?

Ratio test is inconclusive there $(L=1)$.

$x=4$: $\displaystyle\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent (p-test)

$x=2$: $\displaystyle\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergent (alt. series test)

We're done: interval of convergence is $[2,4)$ i.e. $2 \le x < 4$

Remark: this series represents the function $\ln(x-2)$

(See 2 lectures from now.)

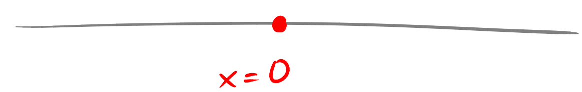
Ex $\displaystyle\sum_{n=1}^{\infty} n!\,x^n = x + 2x^2 + 6x^3 + \cdots$ Power series centered at $a=0$.

Ratio Test: $\displaystyle\frac{|a_{n+1}|}{|a_n|} = \frac{|(n+1)!\,x^{n+1}|}{|n!\,x^n|} = \left(\frac{(n+1)!}{n!}\right)|x| = (n+1)|x|$

As $n \to \infty$, $\displaystyle\lim_{n\to\infty} (n+1)|x| = \begin{cases} \lim\limits_{n\to\infty} 0 & \text{if } x=0 \\ \lim\limits_{n\to\infty} (n+1)|x| & \text{if } x \neq 0 \end{cases} = \begin{cases} 0 & \text{if } x=0 \\ \infty & \text{if } x\neq 0 \end{cases}$

So this series <u>converges</u> for $x=0$, <u>diverges</u> if $x \neq 0$.

Radius of conv. $R=0$.



Ex $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ ("Bessel function")

Ratio test: $\displaystyle\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{|x|^{2n}}$

$$= \frac{|x|^2}{4} \cdot \left[\frac{n!}{(n+1)!}\right]^2 = \frac{|x|^2}{4} \cdot \left(\frac{1}{n+1}\right)^2 \longrightarrow 0 \text{ as } n\to\infty$$

$L = 0 < 1$, so series converges, for <u>all</u> x.

Radius of conv. is $R = \infty$

Interval of conv. is $(-\infty, \infty)$