Q: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(\ln k)^2+3}$ Convergent or not?

A: As $k \to \infty$, $\underset{\substack{\downarrow\\ \infty}}{k}(\underset{\substack{\downarrow\\ \infty}}{\ln k})^2 \to \infty$ also. So, $k(\ln k)^2 \overset{\text{" is much bigger than"}}{\gg} 3$ as $k \to \infty$.

So, we might guess that $\displaystyle\sum \underbrace{\frac{1}{k(\ln k)^2+3}}_{a_k}$ behaves like $\displaystyle\sum \underbrace{\frac{1}{k(\ln k)^2}}_{b_k}$

i.e. use Limit-Comparison Test:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k(\ln k)^2}{k(\ln k)^2+3} = \lim_{k\to\infty} \frac{\overset{\nearrow 1}{1}}{\underset{\swarrow 1}{1} + \underbrace{3/k(\ln k)^2}_{\to 0}} = 1$$

And $\sum b_k$ is easier: can use Integral Test (NB: $\frac{1}{x(\ln x)^2}$ is decreasing function)

$$\int_1^\infty \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_1^{\ln t} \frac{du}{u^2}$$

$u = \ln x$
$du = \frac{dx}{x}$

$$= \lim_{t\to\infty} \frac{1}{u}\Big]_1^{\ln t}$$

$$= \lim_{t\to\infty} \Big(\underset{\substack{\downarrow\\ 0}}{\frac{1}{\ln t}} - \underset{\substack{\downarrow\\ -1}}{1}\Big)$$

which exists, so the $\int$ converges

so the $\sum$ also converges

Remark: The same kind of idea works for many similar series:

For $\frac{1}{k^2 \ln k + 3}$ use limit-comparison to $\frac{1}{k^2 \ln k}$ (conv, by Integral Test)

For $\frac{3k}{6 \ln k + 5}$ use limit-comparison to $\frac{3k}{6 \ln k}$ (diverges, by Divergence Test)

For $\frac{8n}{2n^2 \ln n + 7}$ use limit-comparison to $\frac{8n}{2n^2 \ln n} = \frac{4}{n \ln n}$ (div, by Int Test)