Lecture 1 | 20 Jan 2010

Welcome! M408L, Spring 2010. Integral Calculus.

1st day handout: http://www.ma.utexas.edu/courses/m408l/handout.php

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Lectures MWF 3-4p, WRW 102. [Begin at 3:00, end at 3:50 ± 2 min.]

Discussion TTh
56585 2-3p WRW 113
56590 3:30-4:30p RLM 5.124
56595 5-6p RLM 5.116

Office hours: me MF 1:30-2:30p RLM 9.134
James MW 11:00a-12:30p RLM 10.146

Broadly the same format as 408K:

- Homework via QUEST at https://quest.cns.utexas.edu/ 10%
  Due 3am each Mon night / Tue morning
  One extra review assignment, due 3am Friday Jan 29
  Worst 3 (of 15) dropped in grade
  Working together strongly recommended!
- 3 midterm exams (2 hr, evening) 60%
  Feb 23, Apr 6, May 4
- Final exam 30%
  Date unknown until late in the semester!
  Could be anytime during the exam period.

Significantly harder than 408K ⇒ Have to work harder to get the same grade.

Many resources available: Office hours
UT Learning Center (Jester A332A)
Your fellow students!
Textbook

Slides available at http://www.ma.utexas.edu/users/neitzke
(+ on Blackboard, <u>soon</u>)

Antiderivatives

Recall the derivative: $f(x) \rightsquigarrow f'(x)$

Examples:
- $f(x) = x^2 \rightsquigarrow f'(x) = 2x$
- $f(x) = \sin(x^3) \rightsquigarrow f'(x) = 3x^2\cos(x^3)$

Suppose we wanted to "go backwards":

$$\begin{tikzcd} f \arrow[r, bend left, "\text{derivative}"] & f' \arrow[l, bend left, "\text{antiderivative}"] \end{tikzcd}$$

Definition: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Any function $f(x)$ has many antiderivatives!

Ex: $f(x) = x$ has antiderivatives $F(x) = \frac{1}{2}x^2$

$$F(x) = \tfrac{1}{2}x^2 + 7$$
$$F(x) = \tfrac{1}{2}x^2 - 32$$
$$\vdots$$

To get all possible antiderivatives of $f(x)$, first find one particular antiderivative, and then add an arbitrary constant. (usually called C)

Ex:
- $f(x) = \cos x$ has general antiderivative $F(x) = \sin x + C$
- $f(x) = x^n$ " " " $F(x) = \dfrac{1}{n+1}x^{n+1} + C$ $\boxed{\text{EXCEPT when } n=-1}$

Build more complicated examples from these simple ones:

- $f(x) = 9x^2 + 6x^{3/2} - \dfrac{2}{x^4} + \cos x$ $\qquad \dfrac{1}{x^4} = x^{-4}$

has general antiderivative

$$F(x) = 9\left(\tfrac{1}{3}x^3\right) + 6\left(\tfrac{1}{(5/2)}x^{5/2}\right) - 2\left(\tfrac{1}{-3}x^{-3}\right) + \sin x + C$$
$$= 3x^3 + \tfrac{12}{5}x^{5/2} + \tfrac{2}{3}x^{-3} + \sin x + C$$

- $f(x) = \cos 2x$ has general antideriv
  $F(x) = \frac{1}{2}\sin 2x + C$

- Is $F(x) = \frac{4}{3}\ln(1+3x)$ an antiderivative of $f(x) = \dfrac{4}{1+3x}$?

$$F'(x) = \frac{4}{3}\cdot\frac{1}{1+3x}\cdot 3 = \frac{4}{1+3x} \quad \text{— YES}$$

Sometimes we don't want the <u>most general</u> antideriv, we want some <u>specific</u> one:

- What is the function $f(x)$ which has $f'(x) = 4x+7$?
  <u>AND</u> $f(1) = 6$

Since $f'(x) = 4x+7$, $\quad f(x) = 2x^2 + 7x + C$

Since $f(1) = 6$,

$$2 + 7 + C = 6$$
$$9 + C = 6$$
$$C = -3$$

So $\underline{f(x) = 2x^2 + 7x - 3}$

<u>Why care</u> about antiderivatives?

A Standard reason:

$$\begin{tikzcd}
\text{position } s(t) \arrow[r, bend left, "\text{derivative}"] & \text{velocity } v(t) \arrow[l, bend left, "\text{antideriv.}"] \arrow[r, bend left, "\text{derivative}"] & \text{acceleration } a(t) \arrow[l, bend left, "\text{antideriv.}"]
\end{tikzcd}$$

- A train accelerates with constant accel. $a(t) = 4\ ft/s^2$

  At time $t=0$ it has velocity $v(t=0) = 100\ ft/s$

  and position $s(t=0) = 0\ ft.$

  How far does it go in 20 s? $s(t=20\ s) = ?$

$$\left.\begin{array}{l} a(t) = 4 \\ v(t) = 4t + C \\ v(t=0) = 100 \end{array}\right] \Longrightarrow C = 100$$

so $v(t) = 4t + 100$

$$\left.\begin{array}{l} s(t) = 2t^2 + 100t + D \\ s(t=0) = 0 \end{array}\right] \Rightarrow D = 0$$

so $s(t) = 2t^2 + 100t$

so $s(t=20) = 2(20^2) + 100(20) = \underline{\underline{2800\ ft}}$

Lecture 2 | 22 Jan 2010

Housekeeping:

HW02 due 3am 1/26 (this Tue)
HW01 due 3am 1/29 (this Fri)

- Some of the problems are harder than the examples I show in lecture!
- You can (and should) enter your answers into QUEST as you go, rather than all at once when you finish.

Last time

Antiderivative

$$F(x) \underset{\text{antideriv}}{\overset{\text{deriv}}{\rightleftarrows}} f(x) = F'(x)$$

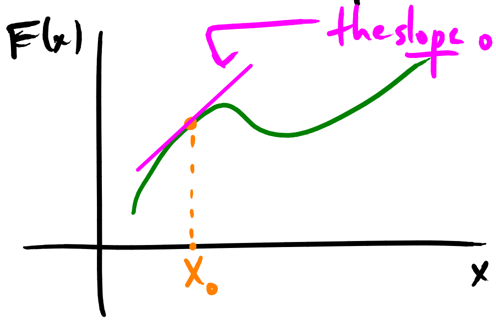
Erratum: The most general antideriv. of $f(x) = x^n$ is $\frac{1}{n+1}x^{n+1} + C$ except when $n = -1$!

The most general antideriv. of $f(x) = x^{-1} = \frac{1}{x}$ is $F(x) = \ln x + C$

More about antideriv (Ch 4.9)

Suppose we don't know a formula for $f(x)$, but we do have its graph. What can we say about its antideriv. $F(x)$?

the slope of this tangent line is $F'(x_0)$

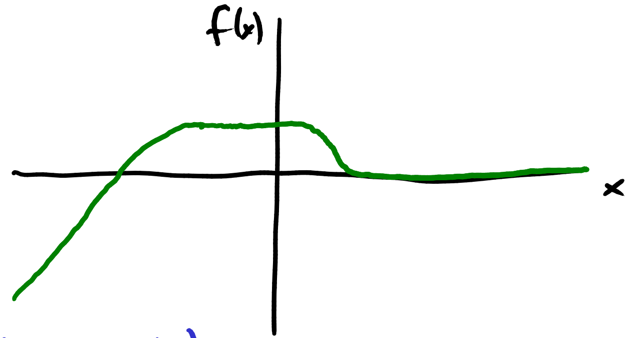


But since F is antideriv. of f,

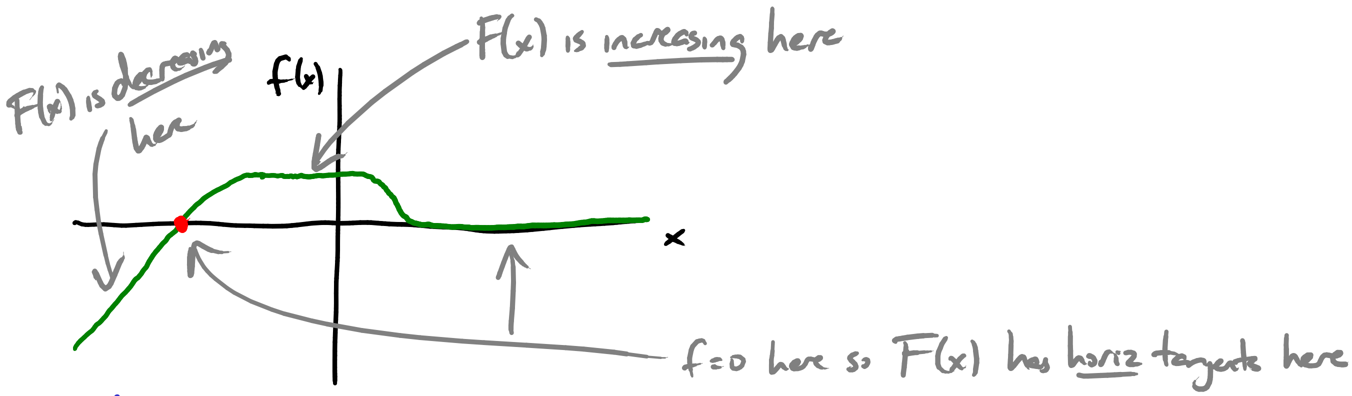
$$F'(x_0) = f(x_0)$$

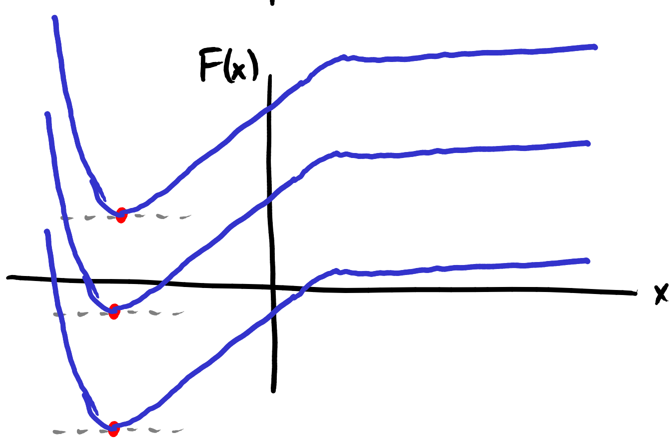
Use this info. about the tangent lines to sketch F.

Ex. Suppose the graph of $f(x)$ looks like



Sketch the graphs of a few possible antideriv. $F(x)$.



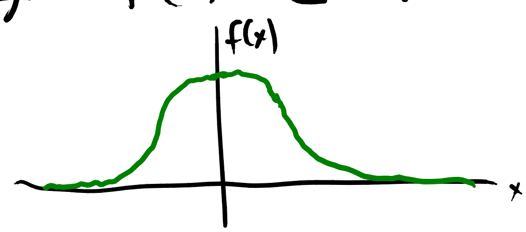


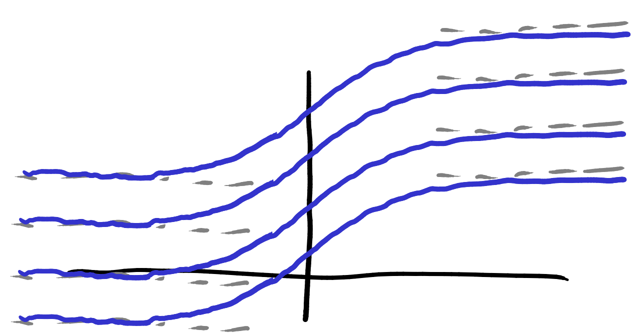
An aside:

Every continuous function $f(x)$ [and even some discontinuous ones...] has an antiderivative $F(x)$.

But sometimes $F(x)$ can't be written in terms of "elementary" functions — even though $f(x)$ can!

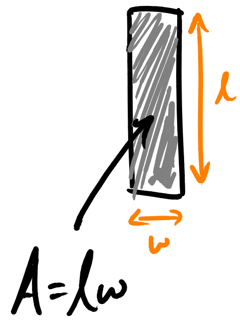
e.g. $f(x) = e^{-x^2}$.



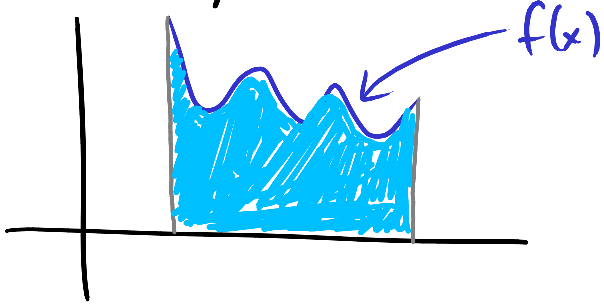


Areas (Ch 5.1)

We all know the areas of simple shapes



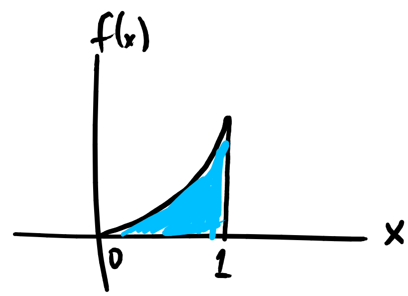
How about a more complicated shape?

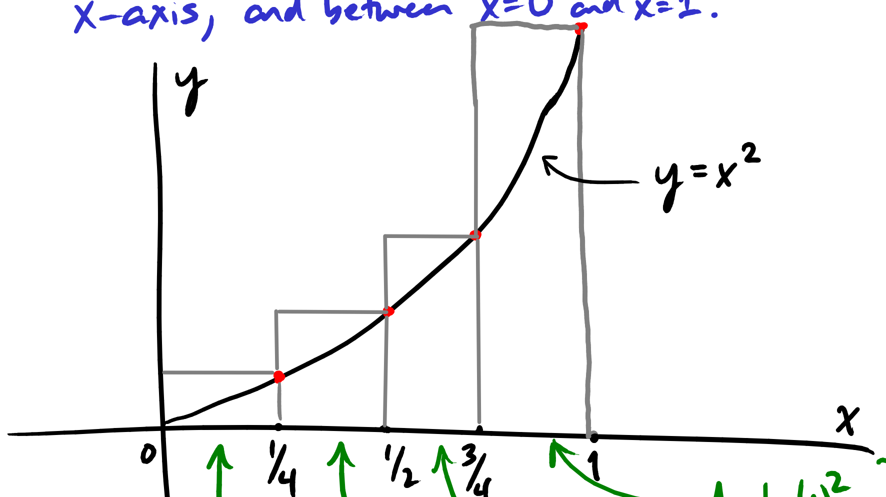


Could try to estimate the area by "counting boxes" (put the figure on graph paper).
Not exact, but if the boxes are very small, it's close.
The exact answer will be the limit of our estimate as the size of the boxes goes to zero.

Ex. Say $f(x)=x^2$

Estimate the area of the region between $y=f(x)$ and the x-axis, and between $x=0$ and $x=1$.





$A=\frac{1}{4}\cdot\left(\frac{1}{4}\right)^2$ $A=\frac{1}{4}\cdot\left(\frac{1}{2}\right)^2$ $A=\frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$ $A=\frac{1}{4}\cdot(1)^2$

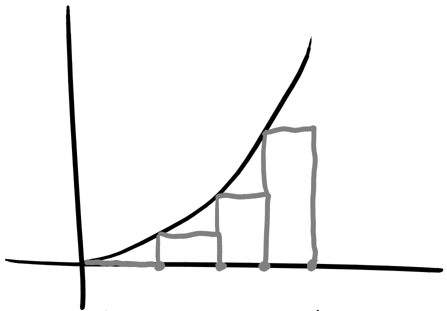
total estimated area

$$A=\frac{1}{4}\left[\left(\frac{1}{4}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{3}{4}\right)^2+1^2\right]$$

simplify $\longrightarrow$ $A = \frac{15}{32}$

This is the "estimated area using 4 rectangles of equal widths and using the right endpoints as the sample pts."

If we used the left endpts



we will get a different estimated area (next time).

Housekeeping:

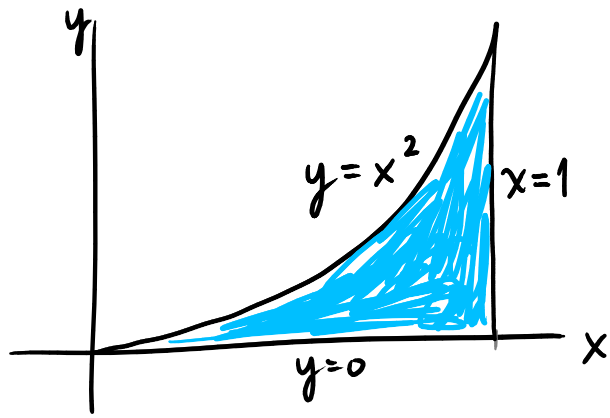
HW02 due 3am 1/26 (tonight!)
HW01 due 3am 1/29 (this Fri)

See me:

ROBERT DERRICK
DOUGLAS MCDOWELL
ANTHONY CARGILE

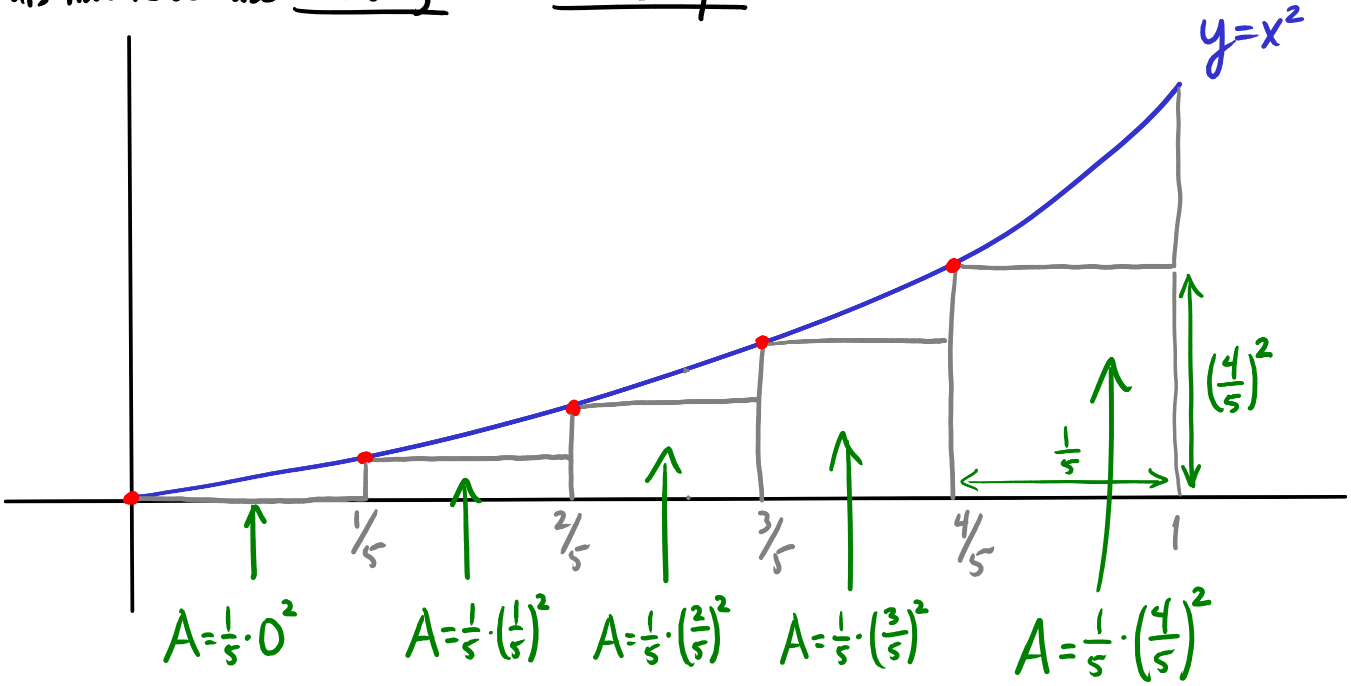
More on areas (Ch 5.1)

Interested in the area under the curve $y=x^2$, between $x=0$ and $x=1$.



Answer depends exactly how you estimate.
Last time we used 4 rectangles and right endpoints as sample points. Got estimate $\frac{15}{32}$.
Write that answer $R_4 = \frac{15}{32}$ (R="right", 4="4 rectangles").

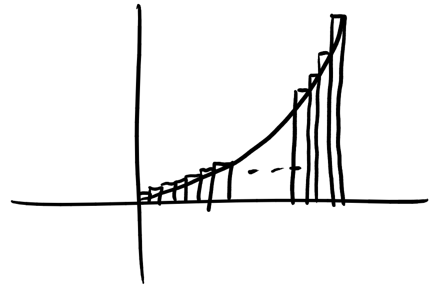
This time let's use 5 rectangles and left endpoints:



Total area $L_5 = \frac{1}{5}\cdot\left(0^2+\left(\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2+\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2\right)$

Now suppose we used 100 rectangles. We would get

$$L_{100} = \frac{1}{100} \cdot \left(0^2 + \left(\frac{1}{100}\right)^2 + \left(\frac{2}{100}\right)^2 + \cdots + \left(\frac{99}{100}\right)^2\right) = 0.3283500$$



$$R_{100} = \frac{1}{100} \cdot \left(\left(\frac{1}{100}\right)^2 + \left(\frac{2}{100}\right)^2 + \cdots + \left(1\right)^2\right) = 0.3383500$$

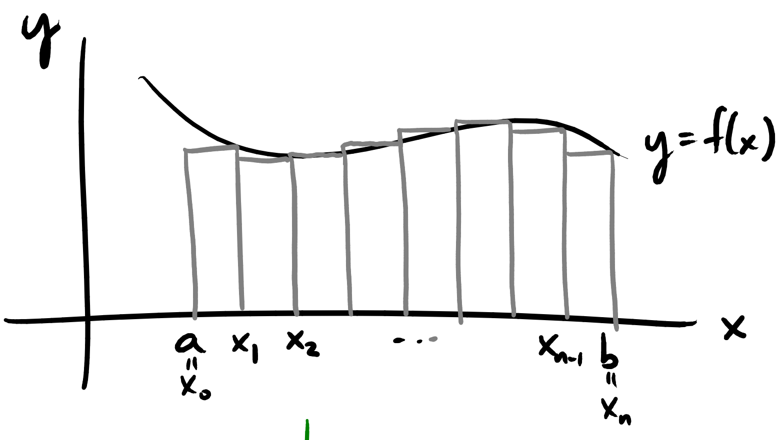
n	L_n	R_n
10	0.2850000	0.3850000
100	0.3283500	0.3385000
1000	0.3328335	0.3338335

Indeed, as $n \to \infty$, both L_n and R_n approach $\frac{1}{3}$: $\left[\text{e.g. } R_n = \frac{(n+1)(2n+1)}{6n^2}\right]$

$$\lim_{n\to\infty} L_n = \frac{1}{3} \qquad \lim_{n\to\infty} R_n = \frac{1}{3}$$

$\frac{1}{3}$ is the <u>exact</u> area under the graph $y = x^2$ between $x=0$ and $x=1$.

For a general function $f(x)$, we can calculate the area similarly:



<u>Width</u> $\Delta x = \frac{b-a}{n}$

<u>Heights</u>: $f(x_1), f(x_2), \ldots, f(x_n)$ [right endpoints] where $x_i = a + i\Delta x$

$\Rightarrow$ <u>area estimate</u> $R_n = \Delta x\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$

Another convenient notation: ("sigma notation")

The symbol $\sum_{i=1}^{n} f(x_i)$ means $f(x_1)+\cdots+f(x_n)$.

Example: Calculate $\sum_{i=1}^{4} i^2$.

$$\sum_{i=1}^{4} i^2 = 1^2+2^2+3^2+4^2 = \underline{\underline{30}}.$$

Example: Write $\frac{2^3}{n}+\frac{4^3}{n}+\frac{6^3}{n}+\cdots+\frac{(2n)^3}{n}$ in sigma notation.

$$\frac{2^3}{n}+\frac{4^3}{n}+\frac{6^3}{n}+\cdots+\frac{(2n)^3}{n} = \underline{\underline{\sum_{i=1}^{n}\frac{(2i)^3}{n}}}$$

In this notation, $R_n = \Delta x \sum_{i=1}^{n} f(x_i)$

And $L_n = \Delta x \sum_{i=1}^{n} f(x_{i-1})$

The actual area is $A = \lim_{n\to\infty} R_n$

<u>or</u> $A = \lim_{n\to\infty} L_n$ (<u>both</u> are the <u>same</u>!)

Example: Let A be the area of the region under the graph of $f(x) = \sin^2 x$ between $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. Using right endpoints as sample points,

- Write a formula for A as a limit.

$a = \frac{\pi}{4}$, $b = \frac{3\pi}{4}$

$$\Delta x = \frac{b-a}{n} = \frac{\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)}{n} = \frac{\pi}{2n}$$

$$x_i = a + i\Delta x = \frac{\pi}{4} + i\frac{\pi}{2n}$$

$$R_n = \Delta x \sum_{i=1}^{n} f(x_i) = \frac{\pi}{2n}\sum_{i=1}^{n} \sin^2\left(\frac{\pi}{4}+i\frac{\pi}{2n}\right)$$

$$A = \lim_{n\to\infty} R_n = \underline{\underline{\lim_{n\to\infty}\left[\frac{\pi}{2n}\sum_{i=1}^{n}\sin^2\left(\frac{\pi}{4}+i\frac{\pi}{2n}\right)\right]}}$$

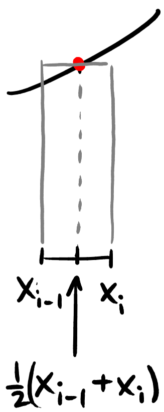
- Estimate A using 3 rectangles.

$$R_3 = \frac{\pi}{6}\sum_{i=1}^{3} \sin^2\left(\frac{\pi}{4} + i\frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}\left[\sin^2\left(\frac{\pi}{4}+\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{4}+\frac{2\pi}{6}\right) + \sin^2\left(\frac{\pi}{4}+\frac{3\pi}{6}\right)\right]$$

$$\approx 1.23885$$

You can also estimate area using other sample points, e.g. the midpoints of the intervals.



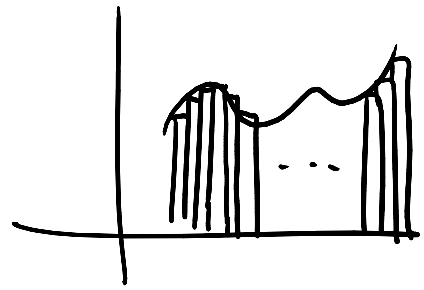
The limit $n \to \infty$ of the estimated area will still be the exact area.

(As long as f is continuous!)

Lecture 4 — 27 Jan 2010

Housekeeping: HW01 due 1/29 3am (this Fri morning)
HW03 due 2/2 3am (next Tue morning)

Last time: computing areas under curves $y = f(x)$ by approximating the region as a union of n rectangles and then taking $n \to \infty$.

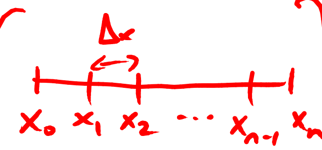


Got $A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i^*)$

x_i^* means the sample point in the i-th interval: could be the left, right, midpoint, etc.

Definite integrals (Ch 5.2)

Definition. Say $f(x)$ is a function defined for $a \le x \le b$. Divide the interval $[a,b]$ into n equal subintervals of equal width Δx, with endpoints $x_0, x_1, \dots, x_n$, where $x_0 = a$ and $x_n = b$.



Pick any "sample points" x_i^* in $[x_{i-1}, x_i]$.

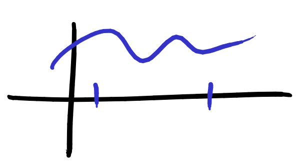
The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

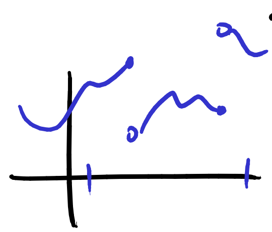
"Riemann sums"

if that limit exists! (If it does, we call f integrable on $[a,b]$.)

[Most f that we encounter in real life are integrable:
e.g. f continuous



or even f with finite # of jumps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Example: Write the definition of $\int_1^3 \sqrt{x}\,dx$.

Divide the interval $[1,3]$ into n equal subintervals with width $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

endpoints $x_0 = 1,\ x_1 = 1+\Delta x,\ x_2 = 1+2\Delta x, \ldots$

$x_i = 1 + i\Delta x = 1 + \frac{2i}{n}$

$$\int_1^3 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{x_i}\,\Delta x$$

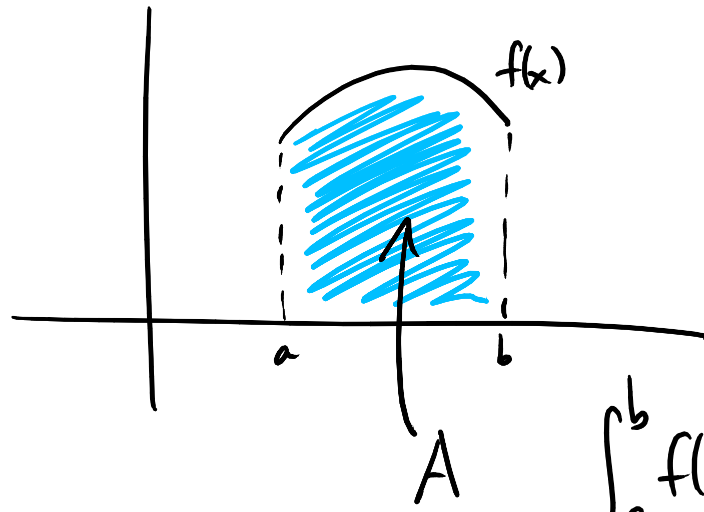
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\frac{2i}{n}}\ \frac{2}{n}$$

$$\left[= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \sqrt{1+\frac{2i}{n}} \right.$$

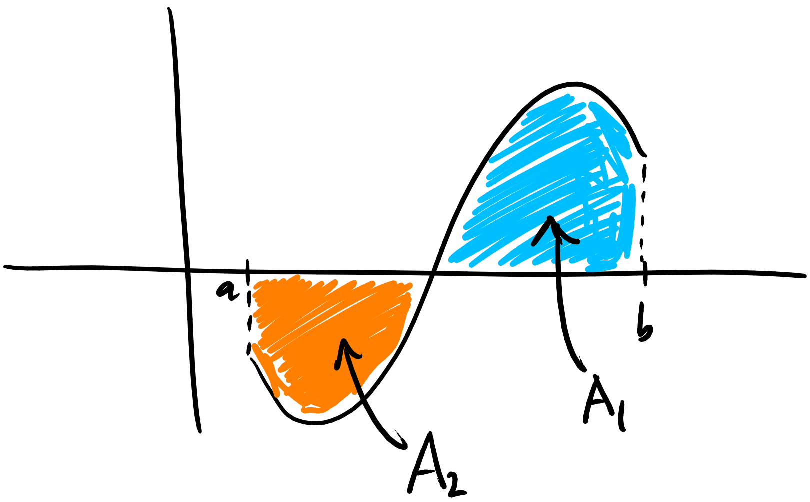
by the general rule

$$\left. \sum_{i=1}^{n} c \cdot a_i = c \sum_{i=1}^{n} a_i \right]$$

Integrals compute areas:



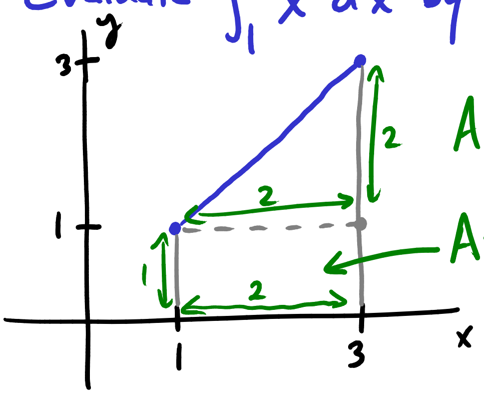
$$\int_a^b f(x)\,dx = A$$



$$\int_a^b f(x)\,dx = A_1 - A_2$$

$$\int_a^b |f(x)|\,dx = A_1 + A_2$$

Example. Evaluate $\int_1^3 x\,dx$ by interpreting it in terms of areas.



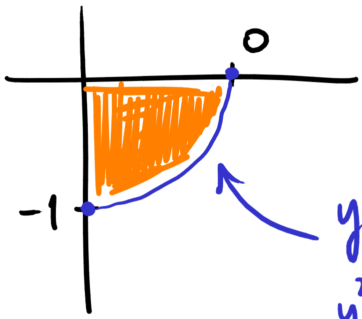
$A = \frac{1}{2}(2\times 2) = 2$

$A = 2\cdot 1 = 2$

total area $2+2=4$

so $\int_1^3 x\,dx = 4$

Example. Evaluate $\int_0^1 -\sqrt{1-x^2}\,dx$ by interp. it in terms of areas.



$y = -\sqrt{1-x^2}$

$y^2 = 1-x^2$

$y^2 + x^2 = 1 \longrightarrow$ circle of radius 1

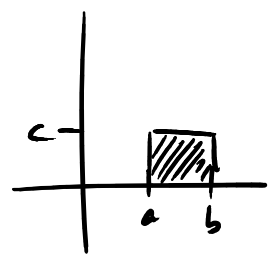
area $= \frac{1}{4}$(area of circle of radius 1) $= \frac{1}{4}(\pi \cdot 1^2) = \frac{\pi}{4}$

So $\int_0^1 -\sqrt{1-x^2}\,dx = \underline{\underline{-\frac{\pi}{4}}}$

minus sign b/c the function is negative on $[0,1]$

A few basic facts about integrals:

1) $\int_a^b c\,dx = c(b-a)$

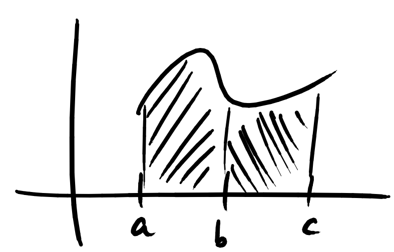


2) $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

4) $\int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



Definition. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

Lecture 5

Reminder: my office hours MF 1:30–2:30 RLM 9.134

Last time: definition of $\int_a^b f(x)\,dx$

Now we learn a much easier way to calculate integrals. (Sec 5.3)

Fundamental Theorem of Calculus I:

If $F(x)=\int_a^x f(t)\,dt$

then $F'(x)=f(x)$. [ie $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.]

Examples. • What is the derivative of $F(x)=\int_{-4}^x \sin t\,dt$?

By FTCI, $F'(x)=\sin x$.

• What is the derivative of $F(x)=\int_4^{x^2}\cos t\,dt$?

[Careful – not just $\cos(x^2)$!]

Apply chain rule: $\frac{d}{dx}\int_4^{x^2}\cos t\,dt$ $\qquad u=x^2$

$$= \frac{d}{dx}\int_4^u \cos t\,dt$$

$$= \frac{du}{dx}\cdot\frac{d}{du}\int_4^u \cos t\,dt$$

$$= 2x\cdot\cos(u)$$

$$= 2x\cdot\cos(x^2)$$

· Suppose $\int_{-1}^{x} f(t)\,dt = \frac{1}{x^2+1}$. What is $f(2)$?

Use FTC I: apply $\frac{d}{dx}$ to both sides.

$$\frac{d}{dx}\int_{-1}^{x} f(t)\,dt = \frac{d}{dx}\frac{1}{x^2+1}.$$

$$f(x) = -\frac{2x}{(x^2+1)^2}$$

$$f(2) = -\frac{4}{25}$$

Ex $\frac{d}{dx}\int_{x}^{5} f(x)\,dx = \frac{d}{dx}\left(-\int_{5}^{x} f(x)\,dx\right) = -f(x)$

Fundamental Theorem of Calculus II:

$\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f.

notation: $F(b)-F(a)$ is also written as $F\Big|_a^b$ (or $F\Big]_a^b$)

[Exercise: try to derive this from FTC I!]

Examples:

· Calculate $\int_0^1 x^2\,dx$.

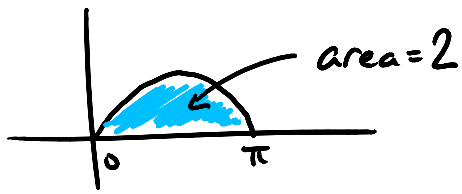
Use FTC II: $F(x) = \frac{1}{3}x^3$ is an antideriv. of x^2, so

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}(1^3) - \frac{1}{3}(0^3) = \frac{1}{3}$$

- Calculate $\int_0^{\pi} \sin x \, dx$.

$F(x) = -\cos x + C$ is an antideriv. of $\sin x$, so

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = (-\cos \pi + C) - (-\cos 0 + C)$$
$$= -(-1) - (-1) + C - C$$
$$= 1 + 1 = \underline{\underline{2}}$$



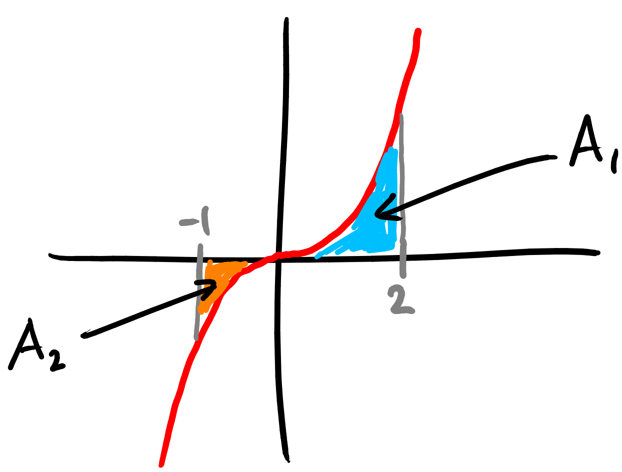
- Calculate $\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta \, d\theta$.

$\sec\theta$ is an antiderivative of $\sec\theta\tan\theta$, so

$$\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta \, d\theta = \sec\theta \Big|_{\pi/4}^{\pi/3} = \sec\tfrac{\pi}{3} - \sec\tfrac{\pi}{4}$$
$$= \underline{\underline{2 - \sqrt{2}}}$$

- Calculate $\int_{-1}^{2} x^3 \, dx$ and interpret it as a <u>difference</u> of <u>areas</u>.

$$\int_{-1}^{2} x^3 \, dx = \frac{x^4}{4} \Big|_{-1}^{2} = \frac{2^4}{4} - \frac{(-1)^4}{4} = \frac{15}{4}.$$



$$A_1 - A_2 = \frac{15}{4}$$

- Calculate $\int_{\pi/6}^{\pi/3} \left(-\frac{3}{\sin^2\theta} + \theta\right) d\theta$.

$$= \int_{\pi/6}^{\pi/3} (-3\csc^2\theta + \theta)\, d\theta$$

$$= 3\cot\theta + \frac{\theta^2}{2} \Big|_{\pi/6}^{\pi/3}$$

$$= \cdots = -2\sqrt{3} + \frac{\pi^2}{24}$$

- Calculate $\int_1^{-2} 3 + u^4\, du$

$$= 3u + \frac{1}{5}u^5 \Big|_1^{-2}$$

$$= \left[3(-2) + \frac{1}{5}(-2)^5\right] - \left[3(1) + \frac{1}{5}(1)^5\right]$$

$$= -6 - \frac{32}{5} - 3 - \frac{1}{5} = -\frac{78}{5}$$

An example that belongs in the previous lecture:

- If $\int_1^3 f(x)\, dx = 4$

  and $\int_3^7 f(x)\, dx = 16$

  what is $\int_1^7 3f(x)\, dx$?

$$= 3\int_1^7 f(x)\, dx$$

$$= 3\left(\int_1^3 f(x)\, dx + \int_3^7 f(x)\, dx\right)$$

$$= 3(4+16) = \underline{\underline{60}}$$

Lecture 6

1 Feb 2010

Last time: Fund^l Theorem of Calculus I, II

I: $\int_a^x f(x)\,dx$ is an antiderivative of $f(x)$.

i.e. $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

II: $$\int_a^b f(x)\,dx = F(b)-F(a) = F\Big|_a^b$$

where F is any antideriv. of f.

Indefinite integrals

Notation: $\int f(x)\,dx$ means any antiderivative of $f(x)$.

Ex: $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x = -\csc x + C$$

Ex Find $\int (10x^4 + 6\sec^2 x)\,dx$

$$= 10\left(\frac{x^5}{5}\right) + 6\tan x + C = \underline{\underline{2x^5 + 6\tan x + C}}$$

Ex Find $\int_0^{\pi/4} (10x^4 + 6\sec^2 x)\,dx$

$$= 2x^5 + 6\tan x\Big|_0^{\pi/4}$$

$$= \left[2\left(\frac{\pi}{4}\right)^5 + 6\tan\left(\frac{\pi}{4}\right)\right] - \left[2(0)^5 + 6\tan(0)\right]$$

$$= \frac{\pi^5}{512} + 6 - 0$$

$$= \frac{\pi^5}{512} + 6$$

Ex Find $\int u^{2/3}\, du$. $\quad n = \frac{2}{3} \quad n+1 = \frac{5}{3}$

$$= \frac{3}{5}u^{5/3} + C$$

Ex Find $\int_1^8 u^{2/3}\, du$.

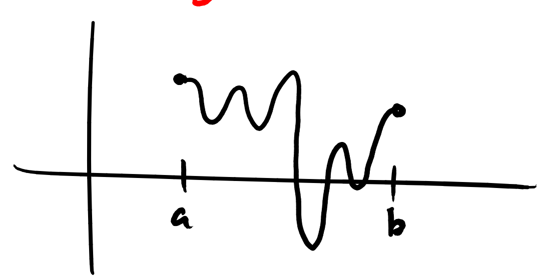
$$= \frac{3}{5}u^{5/3}\Big|_1^8 = \frac{3}{5}\left(8^{5/3} - 1^{5/3}\right)$$

$$= \frac{3}{5}(32-1) = \frac{93}{5}$$

Net change (Ch 5.4)

$F' =$ the rate of change of F.

$$\int_a^b F'(x)\, dx = F(b) - F(a) = \text{net change of } F \text{ over } [a,b].$$



e.g. Water flows into a reservoir at the rate $(10t+6)$ ft^3/s (t in seconds)

The reservoir contains 400 ft^3 of water at $t = 0$.

How much does it have at time $t = 10s$?

The net change from $t=0$ to $t=10$ is

$$\int_0^{10} (10t+6)\, dt = 5t^2 + 6t \Big|_0^{10}$$

$$= \left[5(10^2) + 6(10)\right] - \left[5(0^2) + 6(0)\right]$$

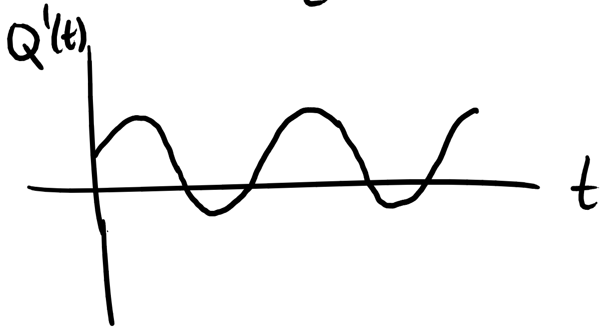
$$= 560 \text{ ft}^3$$

So the amount of water at $t = 10s$ is $400 + 560 = \underline{\underline{960 \text{ ft}^3}}$

e.g. A rechargeable battery is connected to a load that can charge or discharge it. The current flowing into the battery is

$$Q'(t) = \sin(\pi t) + \frac{1}{2}$$

$$\left[Q(t) = \text{the charge of the battery}\right]$$



If the battery starts with 10 units of charge at $t=0$ $[Q(0) = 10]$ how much does it have at $t=6$?

$$Q(t=6) - Q(t=0) = \int_0^6 Q'(t)\, dt$$

$$= \int_0^6 \left(\sin(\pi t) + \frac{1}{2}\right) dt$$

$$= -\frac{1}{\pi}\cos(\pi t) + \frac{t}{2} \Big|_0^6$$

$$= \left(-\frac{1}{\pi}\times 1 + \frac{6}{2}\right) - \left(-\frac{1}{\pi}\times 1 + \frac{0}{2}\right)$$

$$= -\frac{1}{\pi} + 3 + \frac{1}{\pi}$$

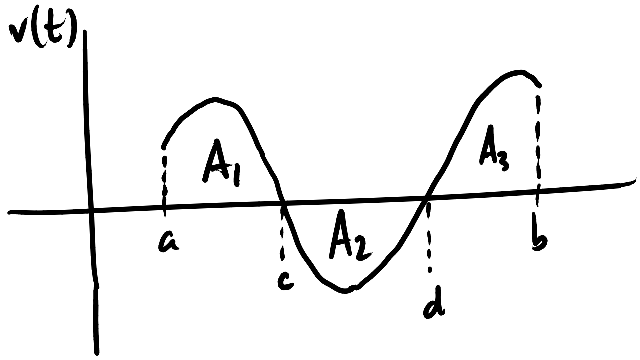
$$= 3$$

$$Q(t=6) = 3 + Q(t=0) = 3 + 10 = \underline{\underline{13}}$$

A standard example of net change: <u>total displacement</u>.

Remember if $s(t)$ = position [along some line]

$s'(t) = v(t)$ velocity



[$v(t) > 0$: $s(t)$ <u>increasing</u>
i.e. moving to the <u>right</u>

$v(t) < 0$: $s(t)$ <u>decreasing</u>
i.e. moving to the <u>left</u>]

<u>Total displacement</u> $s(b) - s(a) = \int_a^b v(t)\, dt = A_1 + A_3 - A_2$

<u>Total distance</u> $A_1 + A_2 + A_3 = \int_a^b |v(t)|\, dt$

$$= \int_a^c v(t)\, dt + \int_c^d -v(t)\, dt + \int_d^b v(t)\, dt$$

<u>Ex</u> A particle moves along a line with $v(t) = t^2 - t - 6$ m/s. (t in sec)
from time $t=1$ to $t=4$.

a) What is the total <u>displacement</u> of the particle?

$$\Delta s = s(4) - s(1) = \int_1^4 v(t)\, dt = \int_1^4 t^2 - t - 6\, dt$$

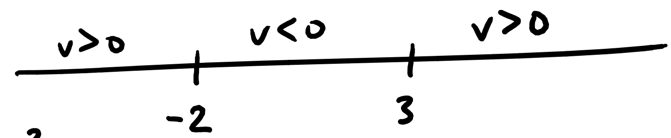
$$= \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_1^4$$

$$= -\frac{9}{2} \text{ m} \qquad \left(\text{i.e. } \frac{9}{2} \text{ m to the } \underline{\text{left}}\right)$$

b) What is the total <u>distance</u> the particle covers?

$$\int_1^4 |v(t)|\, dt$$

$$v(t) = (t-3)(t+2)$$



$$\int_1^4 |v(t)|\, dt = \int_3^4 v(t)\, dt + \int_1^3 -v(t)\, dt$$

$$= \cdots$$

Lecture 7 3 Feb 2010

Last few days: definite and indefinite integrals

Today:

Method of substitution ("u-substitution") (Ch 5.5)

Ex $\int \sqrt{2x-3}\, dx = ?$

Try to get this related to s.t. simpler that we already understand: introduce
$$u = 2x-3$$

Replace x by u everywhere.

$$\int \sqrt{2x-3}\, dx = \int \sqrt{u}\, dx$$

To relate dx to du: $\frac{du}{dx} = 2$, so $du = 2\,dx$
so $\frac{1}{2} du = dx$

$$\text{so } \int \sqrt{u}\, dx = \int \sqrt{u}\, \tfrac{1}{2}\, du$$
$$= \tfrac{1}{2} \int \sqrt{u}\, du$$
$$= \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} + C$$
$$= \tfrac{1}{3} u^{3/2} + C = \underline{\tfrac{1}{3} (2x-3)^{3/2}} + C$$

Ex $\int 7x e^{x^2}\, dx = ?$

Set $u = x^2$.

Then $e^{x^2} = e^u$

and $\frac{du}{dx} = 2x \implies du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$\int 7x e^{x^2}\,dx = 7\int e^u (x\,dx) = 7\int e^u \cdot \tfrac{1}{2}\,du$$

$$= \tfrac{7}{2}\int e^u\,du$$

$$= \tfrac{7}{2} e^u + C$$

$$= \underline{\underline{\tfrac{7}{2} e^{x^2} + C}}$$

Substitution Rule: If $u = g(x)$

then $\int f(g(x))\, g'(x)\,dx = \int f(u)\,du.$

Ex $\displaystyle\int \frac{x^2+16x+8}{\sqrt{\frac{x}{2}+1}}\,dx = ?$

Set $u = \frac{x}{2}+1.$

$\Rightarrow \quad x = 2u-2$

$dx = 2\,du$

$$= \int \frac{(2u-2)^2 + 16(2u-2) + 8}{\sqrt{u}} \cdot 2\,du$$

$$= 2\int \frac{4u^2 - 8u + 4 + 32u - 32 + 8}{\sqrt{u}}\,du$$

$$= 2\int 4u^{3/2} + 24u^{1/2} - 20u^{-1/2}\,du$$

$$= 2\left(4\cdot\tfrac{2}{5}u^{5/2} + 24\cdot\tfrac{2}{3}u^{3/2} - 20\cdot 2u^{1/2}\right) + C$$

$$= \tfrac{16}{5}u^{5/2} + 32u^{3/2} - 80u^{1/2} + C$$

and substitute back $u = \frac{x}{2}+1$ ...

[final result $= \underline{\underline{\tfrac{4}{5}\sqrt{\tfrac{x}{2}+1}\,\left(x^2+24x-56\right)}}$]

Substitution for definite integrals: Remember to transform the endpoints!

Ex $\displaystyle\int_0^{\pi/2} \sin(2x)\,dx$

$u = 2x$

$\dfrac{du}{dx} = 2 \Rightarrow du = 2\,dx$

$dx = \frac{1}{2}\,du$

endpoints: $x=0$ becomes $u=0$

$x=\frac{\pi}{2}$ becomes $u=\pi$

$$= \int_0^{\pi} \sin(u)\cdot \frac{1}{2}\,du$$

$$= \frac{1}{2}\Big(-\cos(u)\Big)\Big]_0^{\pi}$$

$$= \frac{1}{2}\Big[[-(-1)]-[-1]\Big]$$

$$= \frac{1}{2}(1+1) = \underline{\underline{1}}$$

Ex $\displaystyle\int_{\pi/3}^{\pi/2} \cos 3x \; e^{\sin 3x}\,dx$

$u = \sin 3x$

$\dfrac{du}{dx} = 3\cos 3x$

$du = 3\cos 3x\,dx$

$dx = \dfrac{du}{3\cos 3x}$

Limits: $x = \frac{\pi}{3}$ is $u = \sin \pi = 0$

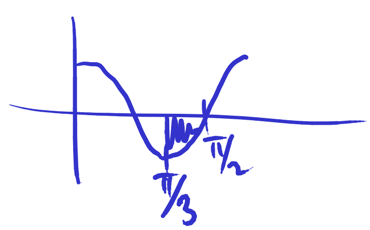
$x = \frac{\pi}{2}$ is $u = \sin\frac{3\pi}{2} = -1$

$$= \int_0^{-1} \cos 3x \; e^{u}\,\frac{du}{3\cos 3x}$$

$$= \frac{1}{3}\int_0^{-1} e^u\,du$$

$$= \frac{1}{3}\Big[e^u\Big|_0^{-1}\Big]$$

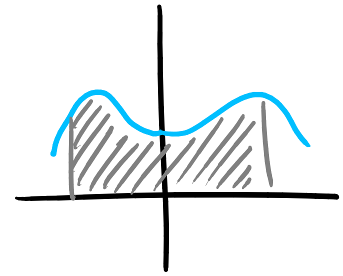
$$= \underline{\underline{\frac{1}{3}\left[e^{-1}-1\right]}}$$



Integrals of symmetric functions:

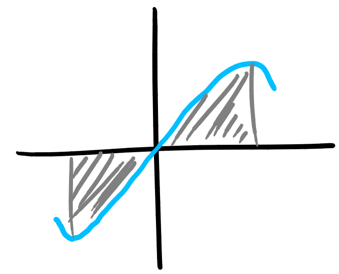
a) If f is even $\qquad f(x) = f(-x)$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



b) If f is odd $\qquad f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = 0$$



Ex

$$\int_{-0.154}^{0.154} \frac{\overbrace{(\tan x)}^{\text{odd}}\overbrace{\left(x^6+29x^4+\frac{105}{3}x^2+981.2\right)}^{\text{even}}}{\underbrace{x^{12}+77x^6+\cos(384x)}_{\text{even}}}\,dx = 0$$

$$\left[\frac{\text{odd}\cdot\text{even}}{\text{even}} = \text{odd}\right]$$

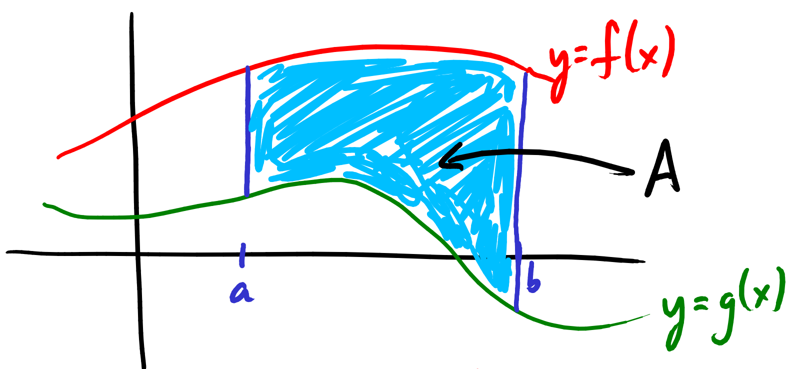
Lecture 8 | 5 Feb 2010

Housekeeping:

Notes available at http://www.ma.utexas.edu/users/neitzke
(+ first day handout)

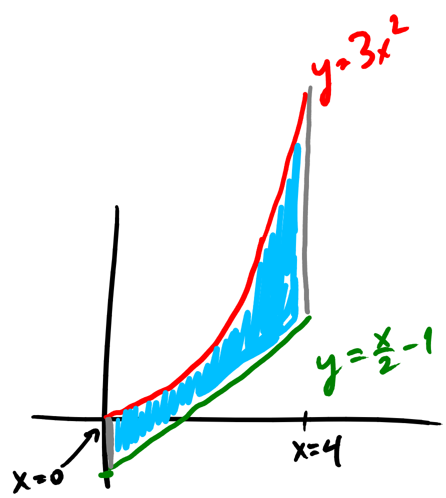
Areas between curves (Ch 6.1)

Two curves $y=f(x)$ and $y=g(x)$. Suppose $f(x)>g(x)$ for x in $[a,b]$.



The area A is given by $\int_a^b (f(x)-g(x))\,dx$.

Example. Find the area between the curves $y=3x^2$ and $y=\frac{x}{2}-1$ from $x=0$ to $x=4$.

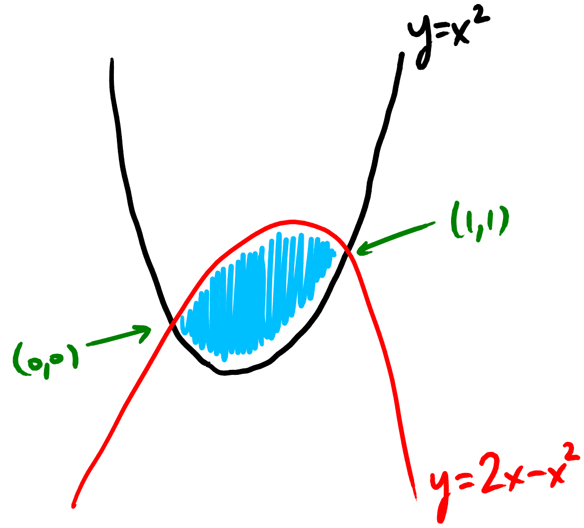


$$A=\int_0^4 (3x^2)-\left(\tfrac{x}{2}-1\right)\,dx$$
$$=\int_0^4 3x^2-\frac{x}{2}+1\,dx$$

$$= x^3 - \frac{1}{4}x^2 + x \Big|_0^4$$

$$= 64 - 0 = \underline{\underline{64}}$$

<u>Example</u>. Find the area of the region between $y = x^2$ and $y = 2x - x^2$.



Start by finding the points of intersection:

$$x^2 = 2x - x^2$$

$$2x^2 - 2x = 0$$

$$2(x)(x-1) = 0$$

So the intersections occur at $x = 0$ $[y=0]$ and $x = 1$. $[y=1]$

$$A = \int_0^1 \left[(2x - x^2) - x^2\right] dx$$

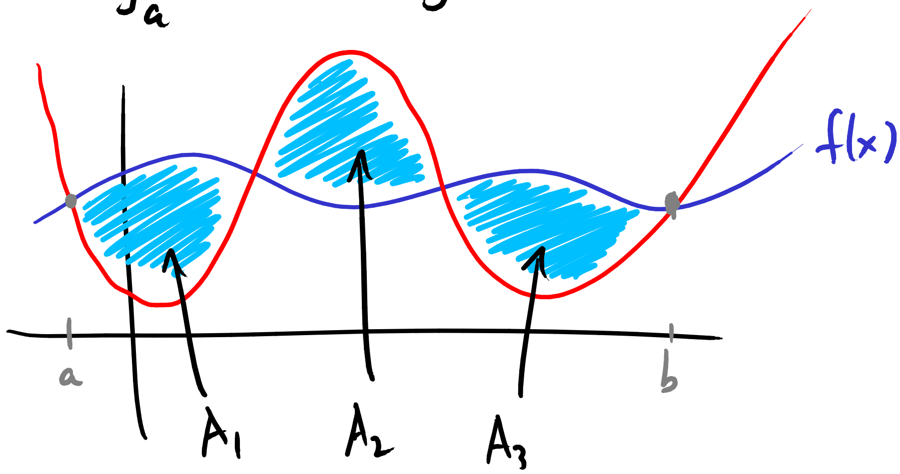
$$= \int_0^1 2x - 2x^2 \, dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1$$

$$= \underline{\underline{\frac{1}{3}}}$$

A rule that finds the area between $y=f(x)$ and $y=g(x)$ no matter which is bigger:

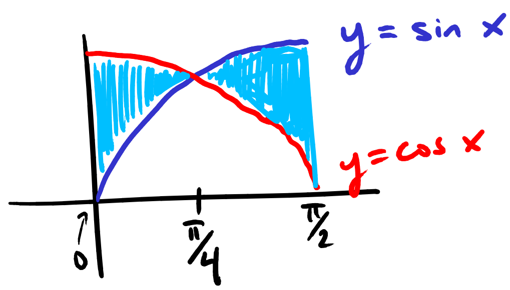
$$A = \int_a^b dx\ \left|f(x)-g(x)\right|$$



$$A = \int_a^b \left|f(x)-g(x)\right| dx = A_1 + A_2 + A_3$$

To actually calculate this $\int$ of absolute value, usually have to break it up into pieces.

Ex. Find the area of the region between $y=\sin x$ and $y=\cos x$, where x ranges between $x=0$ and $x=\frac{\pi}{2}$.



$$A = \int_0^{\pi/2} \left|\cos x - \sin x\right| dx = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

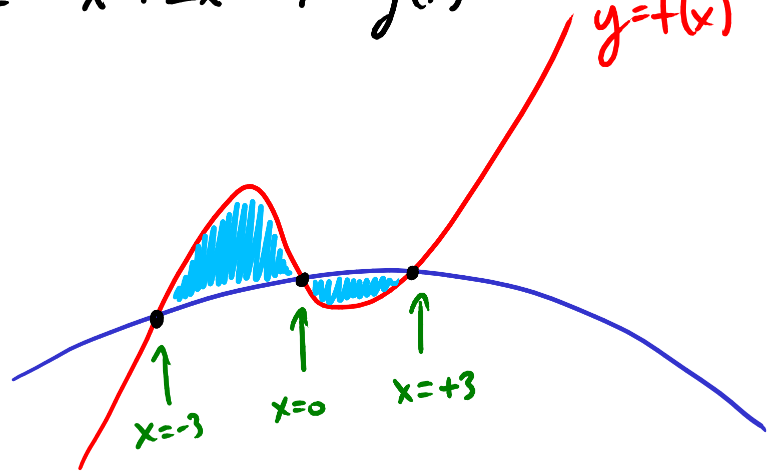
$$= \left(\sin x + \cos x \Big|_0^{\pi/4}\right) + \left(-\cos x - \sin x \Big|_{\pi/4}^{\pi/2}\right)$$

$$= \left(\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) - (0+1)\right) + \left((-0-1) - \left(-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{4}{\sqrt{2}} - 2 = \underline{\underline{2\sqrt{2} - 2}}$$

Ex Find the area of the region between the curves

$$y = x^3 - x^2 - 7x - 4 = f(x)$$
$$y = -x^2 + 2x - 4 = g(x)$$



First, find the points of <u>intersection</u>:

$$x^3 - x^2 - 7x - 4 = -x^2 + 2x - 4$$
$$x^3 - 9x = 0$$
$$x(x+3)(x-3) = 0$$

$\Longrightarrow$ intersections are $\underline{x = 0, -3, +3}$

Area is $\displaystyle\int_{-3}^{3} |f(x) - g(x)|\,dx$

$f(x) - g(x) = x^3 - 9x$

$$= \int_{-3}^{3} |x^3 - 9x|\,dx = \int_{-3}^{3} |x(x+3)(x-3)|\,dx$$

$<0 \quad >0 \quad <0 \quad >0$

$-3 \quad 0 \quad 3$

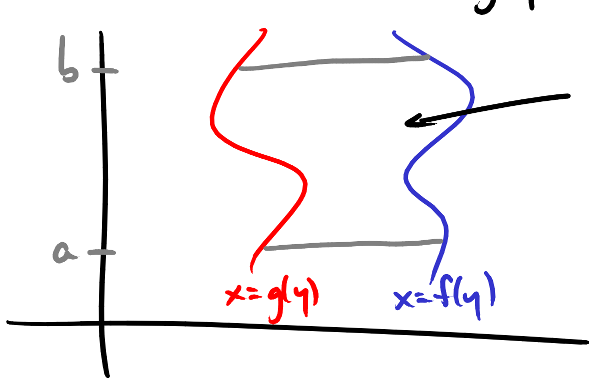
$$= \int_{-3}^{0} x^3 - 9x\,dx + \int_{0}^{3} 9x - x^3\,dx$$

$$= \frac{81}{4} + \frac{81}{4}$$

$$= \frac{81}{2}$$

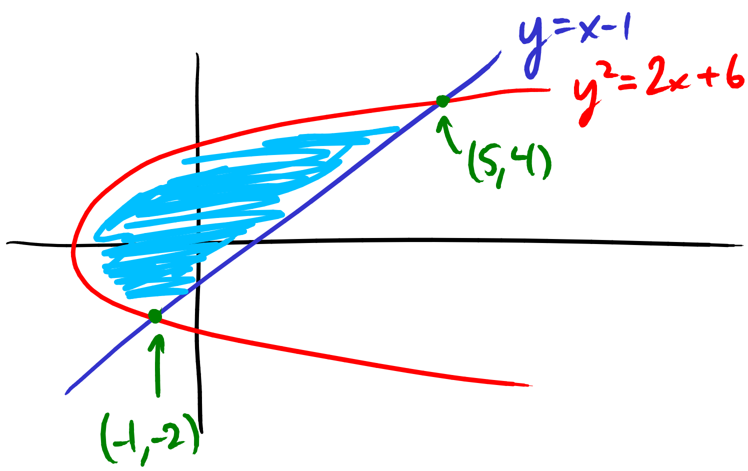
Can also consider two curves $x = f(y)$

$x = g(y)$



$$A = \int_a^b |f(y) - g(y)|\, dy$$

Ex Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



Write the curves as $x = y + 1 = f(y)$

$x = \frac{1}{2}y^2 - 3 = g(y)$

$$A = \int_{-2}^{4} |f(y) - g(y)|\, dy = \int_{-2}^{4} (y+1) - \left(\tfrac{1}{2}y^2 - 3\right) dy = \underline{\underline{18}}$$

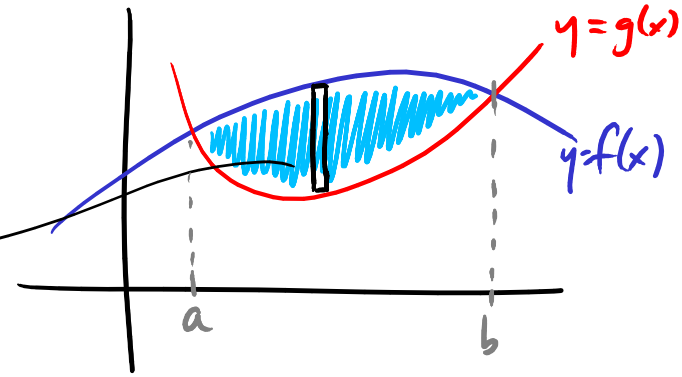
Lecture 9

8 Feb 2010

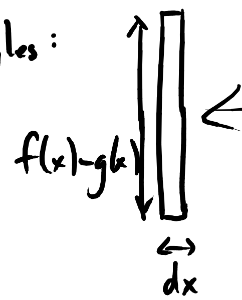
Housekeeping:

1st midterm: 23 Feb 7–9pm WEL 1.316

Last time: Areas between curves



Break into rectangles:

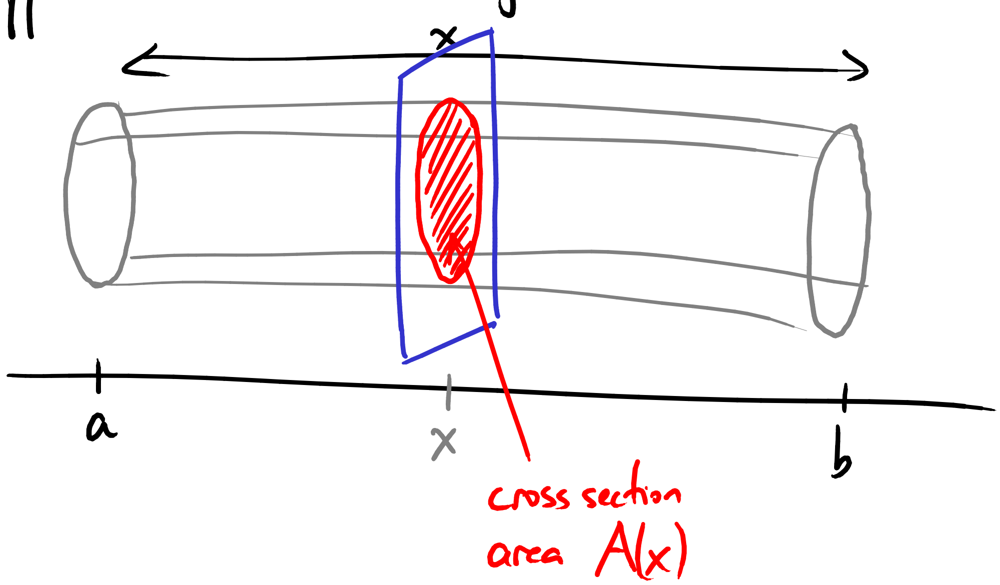


rectangle has $A = (f(x) - g(x))\,dx$

sum them up: total area $= \int_a^b (f(x) - g(x))\,dx$

Volumes (Ch 6.2)

Suppose we have some 3-d object and want to find its volume.

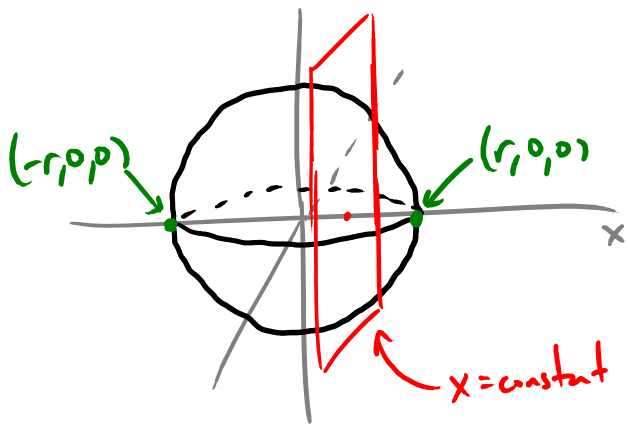


Chop the volume into slices: each slice has volume $A(x)\,dx$

Total volume: add up the slices —

$$V = \int_a^b A(x)\,dx$$

Ex Calculate the volume of a sphere of radius r.



Slice the sphere by planes $x = \text{constant}$.

Sphere is $x^2+y^2+z^2 \leq r^2$

At fixed value of x:

this is $\left(y^2+z^2 \leq r^2 - x^2\right)$ inside y-z plane.

That's the inside of a circle with radius $\sqrt{r^2-x^2}$.

So the cross sections are circles, with area $\pi(r^2-x^2) = A(x)$.

So the volume is

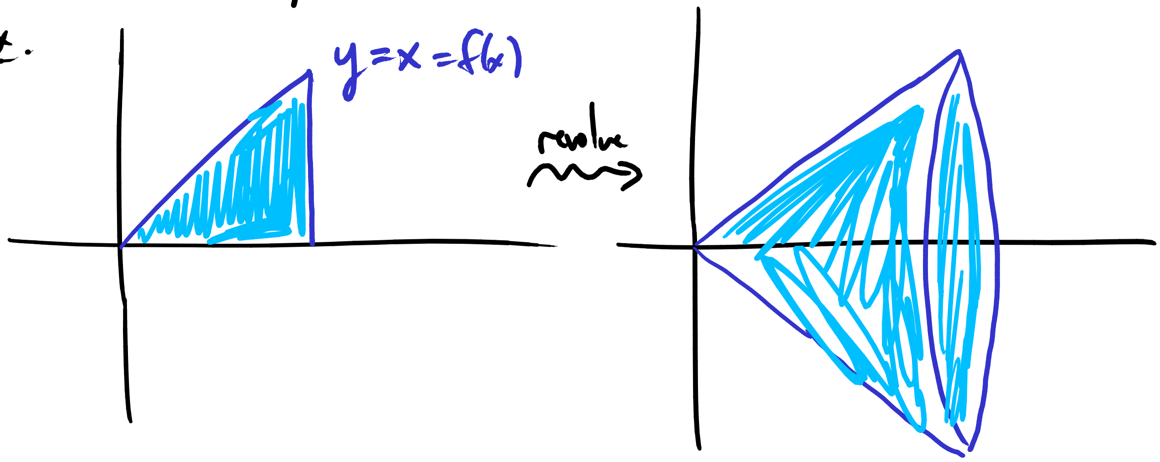
$$V = \int_{-r}^{r} dx\, A(x) = \int_{-r}^{r} \pi(r^2-x^2)\,dx$$

$$= \pi\left(r^2x - \frac{x^3}{3}\right)\Big|_{-r}^{r}$$

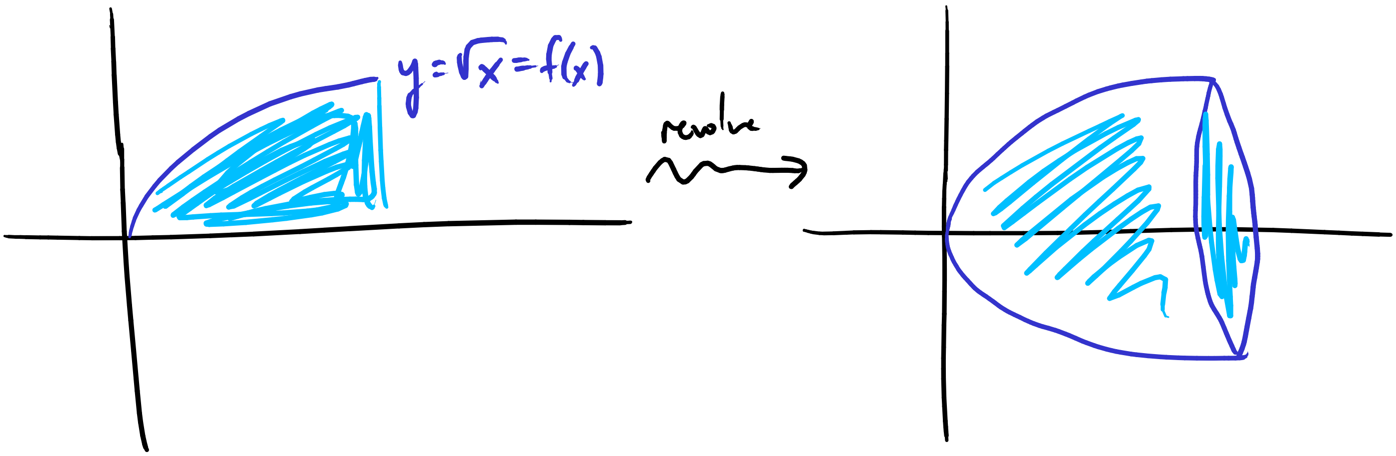
$$= \frac{4}{3}\pi r^3$$

A common class of solid: "solid of revolution" — take the region under some graph and revolve it around, say, the x-axis.

Ex.



cone



The cross-sectional area in this case is just $A(x) = \pi f(x)^2$.
(Because the cross section is a circle with radius = $f(x)$.)

Ex Find the volume of a solid obtained by revolving the area under $y = \sqrt{x}$ around the x-axis, with x from 0 to 2.

$$V = \int_0^2 dx\, A(x) = \int_0^2 dx\, \pi(\sqrt{x})^2 = \int_0^2 dx\, \pi x = 2\pi$$

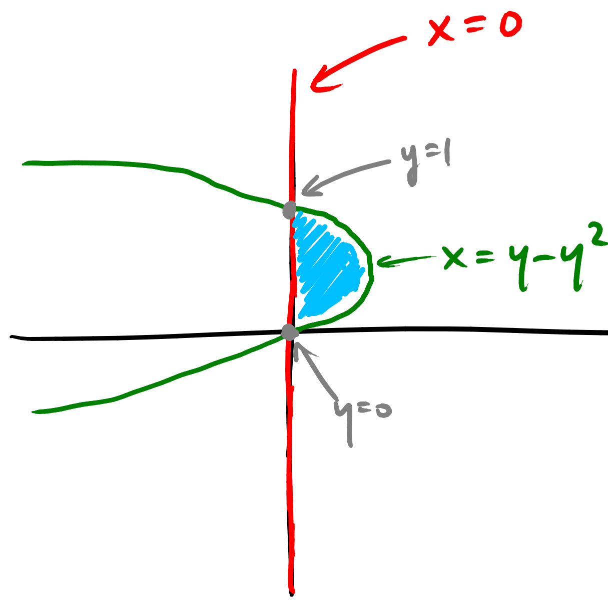
Can also revolve around, say, the y-axis.

Ex Find the volume of a solid obtained by revolving the region between

$$x = y - y^2$$

and $x = 0$

around the y-axis.



Intersections: $y - y^2 = 0$

$y(1-y) = 0$

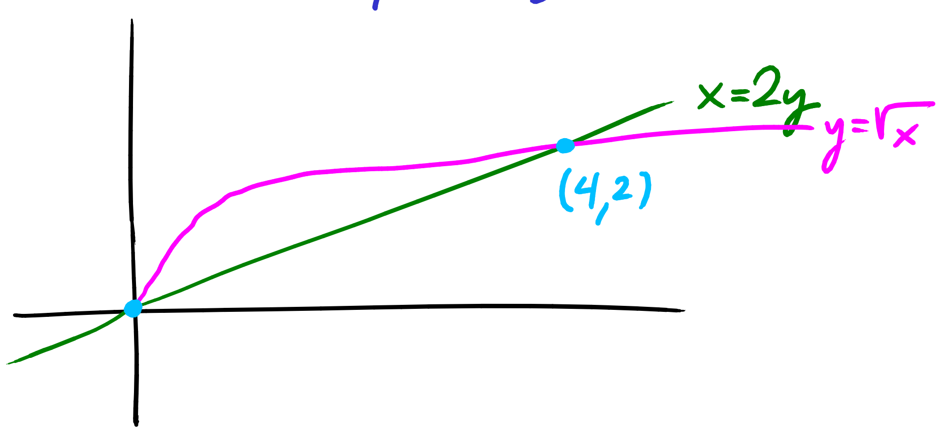
$y=0,\ y=1$

Slices at constant y are circles, with radius $y-y^2$

So $V = \int_0^1 dy\, A(y) = \int_0^1 dy\, \pi (y-y^2)^2 = \underline{\underline{\frac{\pi}{30}}}$

We may also encounter solids which can be sliced into little "washers" (circular disc with a circular hole cut out).

Ex Let R be the region between $y = \sqrt{x}$ and $x = 2y$. Find the volume of the solid obtained by rotating R around the y-axis.



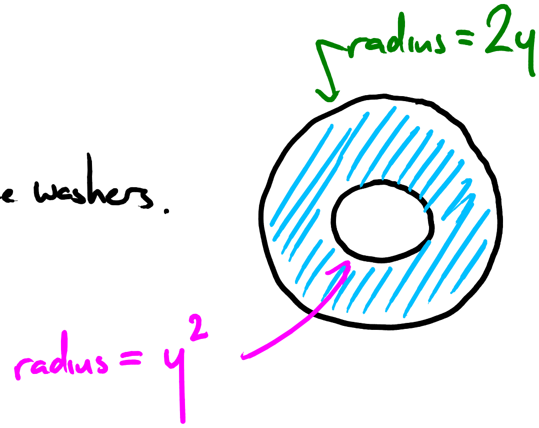
Intersection:

$y = \sqrt{x} \Rightarrow x = y^2$

also $x = 2y$

So $2y = y^2$

$$y^2 - 2y = 0$$
$$y(y-2) = 0 \implies y = 0 \text{ or } y = 2$$



Rotate around y axis: cross sections look like washers.

Radii determined by the distance from the y-axis, i.e. the value of $\underline{\underline{x}}$.

$$\text{Cross section area} = \pi(2y)^2 - \pi(y^2)^2$$

i.e. $A(y) = \pi(4y^2 - y^4)$

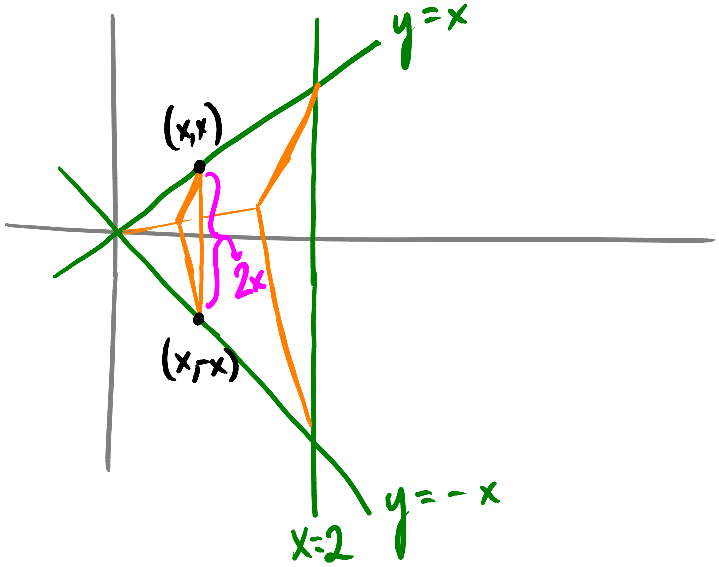
$$V = \int_0^2 A(y)\,dy = \int_0^2 \pi(4y^2 - y^4)\,dy = \underline{\underline{\frac{64}{15}\pi}}$$

Lecture 10 | 10 Feb 2010

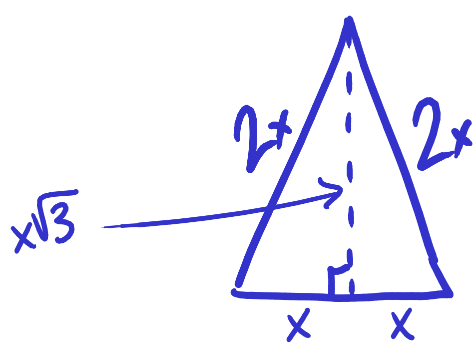
Last time: volumes

$$V = \int_a^b dx\, A(x)$$ ← area of cross section of the solid at fixed x

Ex Calculate the volume of a solid whose base is the region between $y=x$, $y=-x$, and $x=2$, and whose cross sections at fixed x are equilateral triangles.



$$V = \int_0^2 A(x)\, dx \qquad A(x) = \text{area of equilateral } \Delta \text{ with side length } 2x$$

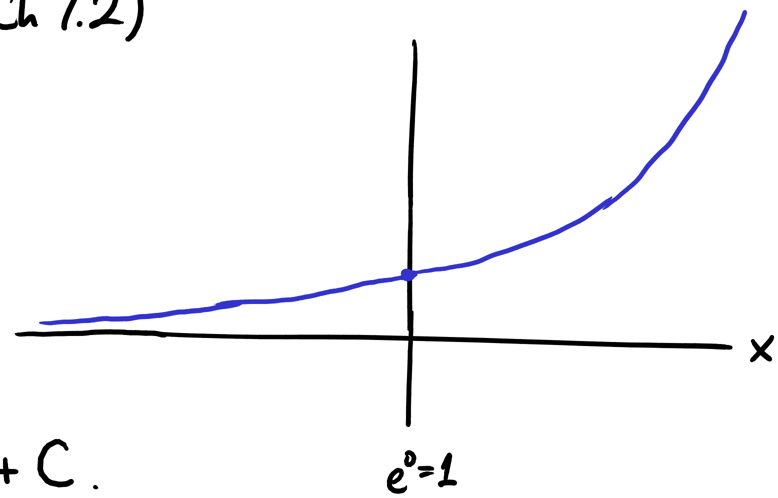


$$\text{area} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}(2x)(x\sqrt{3})$$
$$= x^2\sqrt{3}$$

so $$V = \int_0^2 x^2\sqrt{3}\, dx = \frac{8}{3}\sqrt{3}$$

Exponential functions (Ch 7.2)

$$f(x) = e^x$$



$$\frac{d}{dx} e^x = e^x$$

So also $\int e^x \, dx = e^x + C$.

Ex $\int x^2 e^{x^3} \, dx = ?$

Put $u = x^3$.
Then $du = 3x^2 \, dx$
$dx = \frac{du}{3x^2}$

So $\int x^2 e^{x^3} \, dx = \int x^2 e^u \frac{du}{3x^2} = \frac{1}{3} \int e^u \, du = \frac{1}{3}(e^u + C)$

$= \frac{1}{3}(e^{x^3} + C)$.

Ex $\int e^x \sqrt{1+e^x} \, dx = ?$

Put $u = 1 + e^x$
then $du = e^x \, dx$
so $dx = \frac{du}{e^x}$

So $\int e^x \sqrt{1+e^x} \, dx = \int e^x \sqrt{u} \frac{du}{e^x} = \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C$

$= \frac{2}{3}(1+e^x)^{3/2} + C$

Also remember how to simplify exponents:

$e^a e^b = e^{a+b}$ e.g. $e^{3x^2} e^{-4x} = e^{3x^2 - 4x}$ ..

$(e^a)^b = e^{ab}$ e.g. $(e^{2x+1})^3 = e^{6x+3}$

Ex $\int \frac{(1+e^x)^2}{e^{\frac{1}{2}x}}\,dx = ?$

Multiply out:

$$\int \frac{(1+e^x)^2}{e^{\frac{1}{2}x}}\,dx = \int \frac{1+2e^x+(e^x)^2}{e^{\frac{1}{2}x}}\,dx = \int \frac{1+2e^x+e^{2x}}{e^{\frac{1}{2}x}}\,dx$$

$$= \int e^{-\frac{1}{2}x} + 2e^{\frac{1}{2}x} + e^{\frac{3}{2}x}\,dx \qquad \frac{1}{e^{\frac{1}{2}x}} = e^{-\frac{1}{2}x}$$

Could do by u-substitution on each term separately...

Or, use a shortcut

$$\int e^{bx}\,dx = \frac{1}{b}e^{bx} + C$$

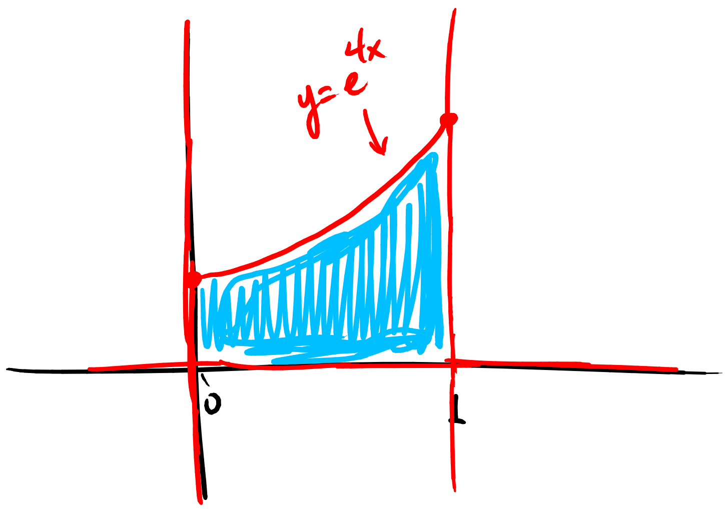
(when b is a constant)

to get

$$= (-2)e^{-\frac{1}{2}x} + (2)2e^{\frac{1}{2}x} + \frac{2}{3}e^{\frac{3}{2}x} + C$$

$$= \underline{\underline{-2e^{-\frac{1}{2}x} + 4e^{\frac{1}{2}x} + \frac{2}{3}e^{\frac{3}{2}x} + C}}$$

Ex Find the volume of the solid obtained by rotating the region bounded by $\begin{bmatrix} y=0 \\ x=0 \\ y=e^{4x} \\ x=1 \end{bmatrix}$ around the x-axis.



$$V = \int_0^1 dx\, A(x) \qquad A(x) = \pi\left(e^{4x}\right)^2 = \pi e^{8x}$$

$$= \int_0^1 dx\, \pi e^{8x}$$

$$= \frac{\pi}{8} e^{8x}\Big|_0^1 = \underline{\underline{\frac{\pi}{8}\left(e^{8}-1\right)}}$$

A caution about exponents:

We have 2 different rules

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} e^x = e^x \qquad [\underline{\text{not}}\ x e^{x-1}!]$$

What about something like $\frac{d}{dx}(x^x)$?

Remember $x = e^{\ln x}$. So $x^x = (e^{\ln x})^x = e^{x \ln x}$

$$\begin{aligned}\frac{d}{dx}(e^{x\ln x}) &= e^{x\ln x}\left[\frac{d}{dx} x \ln x\right]\\ &= e^{x\ln x}\left[\ln x + \frac{x}{x}\right]\\ &= e^{x\ln x}\left[\ln x + 1\right]\\ &= x^x\left[\ln x + 1\right]\end{aligned}$$

Logarithms: (Ch 7.4)

$$e^{\ln x} = x$$

Rules for simplifying:

$$\ln a + \ln b = \ln(ab)$$
$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$
$$k \ln a = \ln a^k$$

Differentiating:

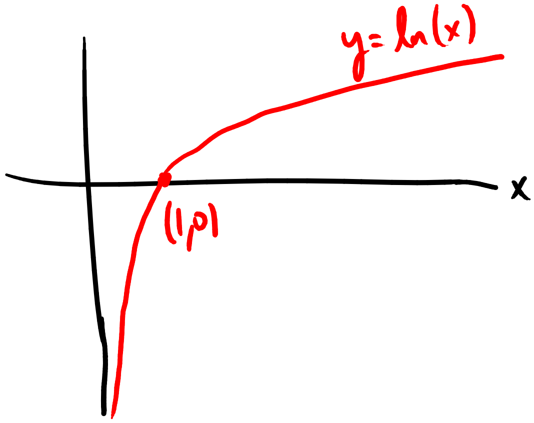
$$\frac{d}{dx} \ln x = \frac{1}{x}$$

NB: $\ln x$ only makes sense when $x > 0$!

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

[Or, can just write $\ln x$ if you know that $x > 0$.]

Ex
$$\int_{-4}^{-2} \frac{1}{x}\, dx = \ln|x| \Big|_{-4}^{-2} = \ln|-2| - \ln|-4|$$
$$= \ln 2 - \ln 4$$
$$= \ln \frac{2}{4}$$
$$= \underline{\underline{\ln \frac{1}{2}}}$$



$\ln(1) = 0$

Ex

$$\int \cot x \, dx = ?$$

$$= \int \frac{\cos x}{\sin x} \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$\left[\frac{du}{dx} = \cos x \right]$

$$= \int \frac{du}{u}$$

$$= \ln |u| + C$$

$$= \ln |\sin x| + C$$

Lecture 11

12 Feb 2010

1^{st} midterm Feb 23 (week from Tue) 7–9pm

Office hours M 1:30–2:30p

F 10:00–11:00a ← CHANGED RLM 9.134

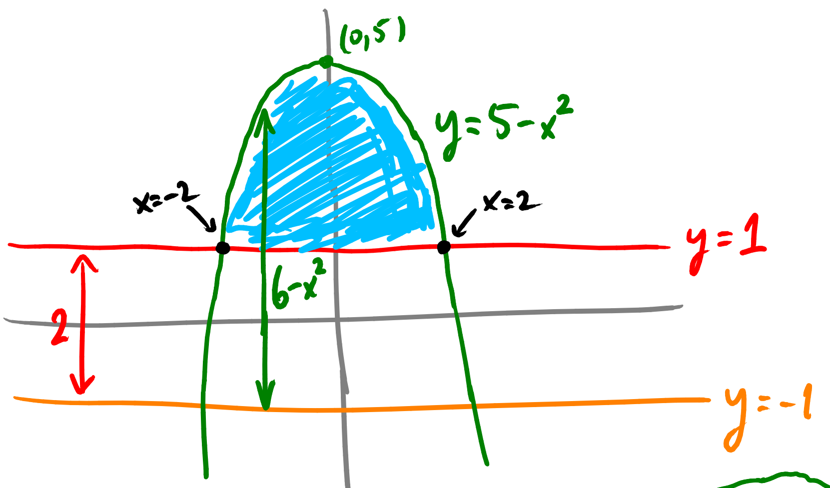
Lecture next M covered by Prof. Daniel Allcock

Ex: Find the volume of solid obtained by rotating the region between $y = 5 - x^2$ and $y = 1$ around the line $y = -1$.

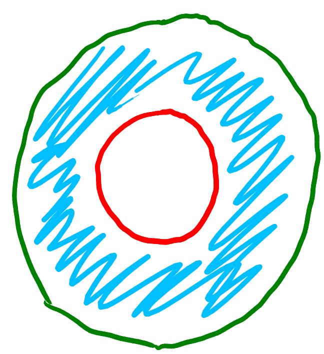
Intersection pts:

$1 = 5 - x^2$

$x = \pm 2$



Cross sections at fixed x look like washers:



radius of inner circle = 2

" " outer circle = $6 - x^2$

$$V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} \pi\left((6-x^2)^2\right) - \pi\left(2^2\right)\,dx$$

$$= \frac{384\pi}{5}$$

Limits of integration and u-substitution

Ex Find $\int_1^e \frac{\ln x}{x}\,dx$.

Take $u = \ln x$: $du = \frac{dx}{x}$

$x\,du = dx$

$x=1 \Rightarrow u = \ln 1 = 0$

$x=e \Rightarrow u = \ln e = 1$

$$\int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}$$

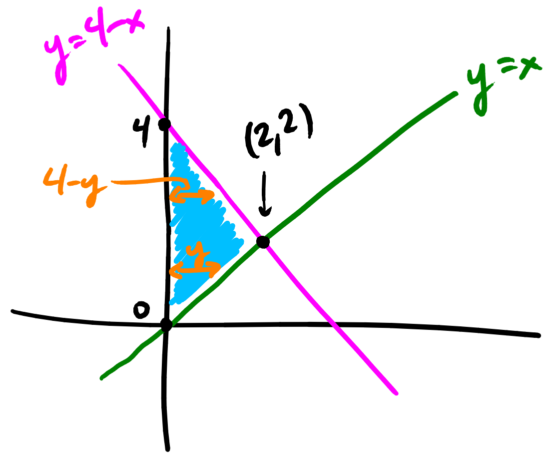
Ex Find the volume of the solid obtained by rotating the region between

$y=x$

$y=4-x$

and the y-axis

around the y-axis.



$y=4-x$

$x=4-y$

$$V = \int_0^4 dy\, A(y) = \int_0^2 dy\, A(y) + \int_2^4 dy\, A(y)$$

$$= \int_0^2 dy\, \pi y^2 + \int_2^4 dy\, \pi(4-y)^2$$

$$= \frac{8\pi}{3} + \frac{8\pi}{3}$$

$$= \frac{16\pi}{3}$$

Ex Find $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$.

Try $u = 1+\cos^2 x$.

$$\frac{du}{dx} = (-\sin x)(2\cos x) \rightarrow du = -2\sin x\cos x\,dx$$
$$= -\sin 2x\,dx$$

So $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = \int \frac{-du}{u} = -\int \frac{1}{u}\,du$

$$= -\ln|u| + C$$
$$= -\ln|1+\cos^2 x| + C$$
$$= -\ln(1+\cos^2 x) + C \quad \left(= \ln \frac{1}{1+\cos^2 x}\right)$$

$$-\ln A = \ln \frac{1}{A}$$

Ex $\displaystyle\int_0^3 \frac{5x^2+10x+2}{10x^2+4}\,dx$

Might first try $u = 10x^2+4$

then $du = 20x\,dx$ ie $dx = \dfrac{du}{20x}$

$$\int_4^{94} \frac{5x^2+10x+2}{u}\,\frac{du}{20x}$$

Looks hard — try splitting the integral up:

$$\int_0^3 \frac{5x^2+2}{10x^2+4}\,dx + \int_0^3 \frac{10x}{10x^2+4}\,dx$$

$$= \int_0^3 \frac{5x^2+2}{2(5x^2+2)}\,dx + \int_0^3 \frac{10x}{10x^2+4}\,dx$$

$$= \int_0^3 \frac{1}{2}\,dx + \int_0^3 \frac{10x}{10x^2+4}\,dx$$

easy $\left(=\frac{3}{2}\right)$

u-subst. $u = 10x^2+4$

gives $\frac{1}{2}\ln\left(\frac{47}{2}\right)$

Ex

$$\int e^x (4+e^x)^3\,dx$$

could just multiply it out (painful!)

But substitute $u = 4+e^x$

$$du = e^x\,dx$$

$$dx = \frac{du}{e^x}$$

$$\longrightarrow \int e^x u^3 \frac{du}{e^x} = \int u^3\,du = \frac{u^4}{4} + C = \frac{(4+e^x)^4}{4} + C$$

Ex $\int_0^3 \frac{5x^2+10x+2}{10x^2+4}\,dx$ $\quad u = 10x^2+4$
$du = 20x\,dx \quad dx = \frac{du}{20x}$

$= \int_0^3 \frac{5x^2+2}{10x^2+4}\,dx + \int_0^3 \frac{10x}{10x^2+4}\,dx$

$= \int_0^3 \frac{5x^2+2}{2(5x^2+2)}\,dx$

$= \int_0^3 \frac{1}{2}\,dx + \int_0^3 \frac{10x}{10x^2+4}\,dx$ $\quad u = 10x^2+4 \quad du = 20x\,dx$
$dx = \frac{du}{20x}$

$= \boxed{\frac{3}{2} + \frac{1}{2}\ln\left(\frac{47}{2}\right)}$

Ex $\int e^x(4+e^x)^3\,dx$ $\quad u = 4+e^x \quad du = e^x\,dx$
$dx = \frac{du}{e^x}$

$\int u^3\,dx$

$\frac{u^4}{4} + C$

$\boxed{\frac{(4+e^x)^4}{4} + C}$

15 Feb 2010

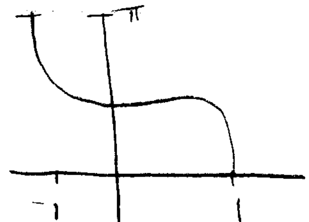
Inverse Trig Functions

$(\tan^{-1}x)' = \frac{1}{1+x^2}$

$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$

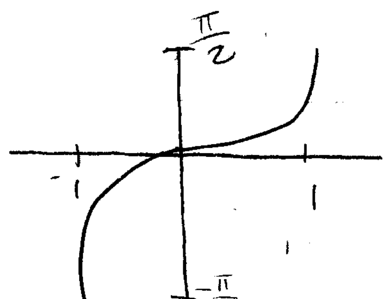
$(\sec^{-1}x)' = \frac{1}{x\sqrt{x^2-1}}$

$\cos^{-1} x$



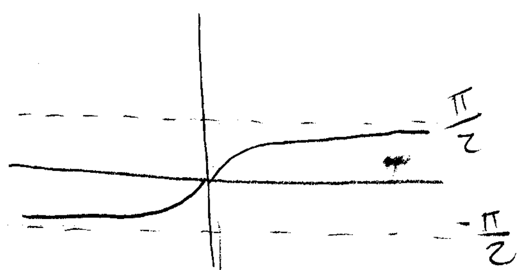
$\cos^{-1} x =$ "the" angle θ whose
cos in x: the θ in $[0, 2\pi]$
so $\cos(\theta) = x$

$\sin^{-1} x$



The θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ give
every value of sin so $\sin\theta = x$

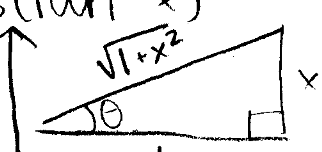
$\tan^{-1} x$



θ is $(-\frac{\pi}{2}, \frac{\pi}{2})$ so $\tan\theta = x$

Ex: $\cos(\tan^{-1} x)$

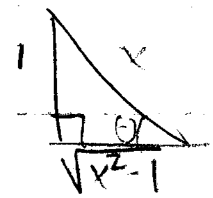
$\tan = \frac{\text{opp}}{\text{adj}}$



$\cos = \frac{\text{adj}}{\text{hyp}}$ $\boxed{\frac{1}{\sqrt{1+x^2}}}$

$\sec(\csc^{-1} x) = \boxed{\dfrac{x}{\sqrt{x^2-1}}}$

$\csc = \dfrac{1}{\sin} = \dfrac{1}{\frac{opp}{hyp}} = \dfrac{hyp}{opp}$



$\sin^{-1}(\sin \frac{3\pi}{4})$

$\frac{3\pi}{4}$ isn't in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so go to θ $\boxed{\frac{\pi}{4}}$ b/c both have side $\boxed{\frac{1}{\sqrt{2}}}$

Integration

$\int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$ $\qquad$ $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

$\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$ $\qquad$ $\int -\dfrac{dx}{1+x^2} = \cot^{-1} x + 1$

$\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{6}{\sqrt{1-t^2}}\, dt = 6\sin^{-1} t \Big]_{1/2}^{\sqrt{3}/2}$

$6\sin^{-1}(\frac{\sqrt{3}}{2}) - 6\sin^{-1}(\frac{1}{2})$

$6(\frac{\pi}{3} - \frac{\pi}{6}) = \boxed{\pi}$

$\int \dfrac{1+x}{1+x^2}\, dx = \int \dfrac{dx}{1+x^2} + \int \dfrac{x\, dx}{1+x^2}$ $\qquad$ $u = 1+x^2$, $du = 2x\, dx$, $x\, dx = \frac{du}{2}$

$\int \dfrac{\frac{du}{2}}{u}$

$\tan^{-1} x + \frac{1}{2} \ln|u| + C$

$\boxed{\tan^{-1} x + \frac{1}{2} \ln|1+x^2| + C}$

$\boxed{\tan^{-1} x + \frac{1}{2} \ln(1+x^2) + C}$

$\int \dfrac{dx}{\sqrt{1-x^2}\, \sin^{-1} x}$ $\qquad$ $u = \sin^{-1} x$ $\qquad$ $du = \dfrac{1}{\sqrt{1-x^2}}\, dx$

$= \int \dfrac{du}{u} = \ln|u| + C = \boxed{\ln|\sin^{-1} x| + C}$

RLM 7.104
Review 3-5 on sunday

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ a is a constant

- Introduce new variable $x^2 = a^2 \cdot u^2$

set $x = au$
$dx = a\,du$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \frac{a\,du}{\sqrt{a^2-a^2u^2}}$

$\frac{a}{\sqrt{a^2}}\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u)+C$

$\boxed{\sin^{-1}\left(\frac{x}{a}\right)+C}$

$\sqrt{a^2}$ $=1$

Ex. $\int \frac{x}{1+x^4}\,dx = \int \frac{x\,dx}{1+(x^2)^2}$ $u = x^2$ $du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$\int \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C$

$= \boxed{\frac{1}{2}\tan^{-1}(x^2)+C}$

16 Feb. 2010

$\int_0^1 \frac{1}{1+x^2}\,dx$ $\arctan x \Big]_0^1$

$\arctan(1) - \arctan(0)$

$\frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$

SOHCAHTOA

$\tan^{-1}(\sqrt{3}) = \boxed{\frac{\pi}{3}}$

$\cos(\arcsin \frac{1}{2})$

$\cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$

$\tan^{-1}(\tan(\frac{3\pi}{4}))$

$\tan^{-1}(-1) = \boxed{-\frac{\pi}{4}}$ in our range

$\tan^{-1}(\tan(\frac{5\pi}{6}))$

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \boxed{-\frac{\pi}{6}}$

Lecture 13 | 17 Feb 2010

Integration By Parts (Ch 8.1)

Product rule:

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x)$$

Take $\int$ of both sides:

$$f(x)\,g(x) = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

$$\Rightarrow \quad \int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

Or: write $u = f(x)$ $\quad v = g(x)$

$du = f'(x)\,dx$ $\quad dv = g'(x)\,dx$

$$\Rightarrow \quad \int u\,dv = uv - \int v\,du$$

Ex Find $\int x \sin(x)\, dx$.

Say $\quad u = x \quad v = -\cos x$

$\quad du = dx \quad dv = \sin(x)\, dx$

Then $\int x \sin(x)\, dx$

$$= \int u\, dv$$

$$= uv - \int v\, du$$

$$= -x\cos x - \int (-\cos x)\, dx$$

$$= \underline{\underline{-x\cos x + \sin x + C}}$$

Ex Find $\int \ln x\, dx$

Try $\quad u = \ln x \quad v = x$

$\quad du = \frac{1}{x} dx \quad dv = dx$

$$\int \ln x\, dx = \int u\, dv$$

$$= uv - \int v\, du = x\ln x - \int x \cdot \frac{1}{x}\, dx$$

$$= x \ln x - \int 1\, dx$$

$$= \underline{\underline{x\ln x - x + C}}$$

Ex Find $\int e^t t^2\,dt$.

Suppose we try $u = e^t$ $\quad v = \frac{1}{3}t^3$

$du = e^t\,dt \quad dv = t^2\,dt$

Then $\int e^t t^2\,dt = \int u\,dv = uv - \int v\,du$

$= e^t \cdot \frac{1}{3}t^3 - \int \frac{1}{3}t^3 e^t\,dt$

$\Longrightarrow$ getting <u>harder</u>! Wrong choice of u, v.

Take $u = t^2 \qquad v = e^t$

$du = 2t\,dt \qquad dv = e^t\,dt$

$$\int e^t t^2\,dt = \int u\,dv = uv - \int v\,du$$

$$= t^2 e^t - \int e^t\,2t\,dt$$

Use int. by parts <u>again</u>: <u>new</u> u, v

$u = 2t \qquad v = e^t$

$du = 2\,dt \qquad dv = e^t\,dt$

So the original $\int$ becomes

$$= t^2 e^t - \left[\int u\,dv\right]$$

$$= t^2 e^t - \left[uv - \int v\,du\right]$$

$$= t^2 e^t - \left[2te^t - \int e^t\,2\,dt\right]$$

$$= \underline{\underline{t^2 e^t - 2te^t + 2e^t + C}} \qquad = \underline{\underline{e^t(t^2 - 2t + 2) + C}}$$

Ex $\int_0^{\pi} t \sin 3t \, dt$

Pick $u = t$ $\quad v = -\frac{1}{3}\cos 3t$

$du = dt$ $\quad dv = \sin 3t \, dt$

$$\int_0^{\pi} t \sin 3t \, dt = \int_0^{\pi} u \, dv = uv \Big|_0^{\pi} - \int_0^{\pi} v \, du$$

$$= (t)\left(-\frac{1}{3}\cos 3t\right)\Big|_0^{\pi} - \int_0^{\pi} \left(-\frac{1}{3}\cos 3t\right) dt$$

$$= (t)\left(-\frac{1}{3}\cos 3t\right)\Big|_0^{\pi} + \frac{1}{3}\left(\frac{1}{3}\sin 3t \Big|_0^{\pi}\right)$$

$$= \left(-\frac{\pi}{3}\cos 3\pi - 0\right) + \frac{1}{9}(0 - 0)$$

$$= \underline{\underline{\frac{\pi}{3}}}$$

Ex $\int e^x \sin x \, dx$

Try $u = e^x$ $\quad v = -\cos x$

$du = e^x dx$ $\quad dv = \sin x \, dx$

$$\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du$$

$$= -e^x \cos x - \int (-\cos x) e^x \, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

Int. by parts again:

$u = e^x$ $\quad v = \sin x$

$du = e^x dx$ $\quad dv = \cos x \, dx$

$$\int e^x \sin x \, dx = -e^x \cos x + \int u \, dv$$

$$\int e^x \sin x \, dx = -e^x \cos x + \left[uv - \int v \, du\right]$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x \, e^x \, dx$$

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = \underline{\underline{\tfrac{1}{2}\left(-e^x \cos x + e^x \sin x\right) + C}}$$

Lecture 14

19 Feb 2010

Housekeeping –

1st Exam Feb 23 7–9p

Covers: antiderivatives thru integration by parts

Review session Sun Feb 21 3–5p RLM 7.104

Int. by parts cont'd (Ch 8.1)

$$\int u\,dv = uv - \int v\,du$$

Sometimes we have to use both subst. and $\int$ by parts.

Ex $\displaystyle\int \cos(\sqrt{x})\,dx = ?$

Try substitution $u = \sqrt{x} = x^{\frac{1}{2}}$.

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad \text{so} \qquad dx = 2\sqrt{x}\,du$$

$$\int \cos(\sqrt{x})\,dx = \int \cos(u)\cdot 2\sqrt{x}\,du$$

$$= \int \cos(u)\cdot 2u\,du$$

Just to avoid confusion, let's change the name of the variable $u \to t$

So now $t = \sqrt{x}$

$$\int \cos(t)\cdot 2t\,dt$$

Int. by parts:

$u = 2t \qquad v = \sin(t)$

$du = 2\,dt \quad dv = \cos(t)\,dt$

Then $\int \cos(t) \cdot 2t = \int u\,dv = uv - \int v\,du$

$= 2t\sin(t) - \int 2\sin(t)\,dt$

$= 2t\sin(t) + 2\cos(t) + C$

Remember $t = \sqrt{x}$:

$$= 2\sqrt{x}\,\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$$

Ex $\displaystyle\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta$

Try $t = \theta^2$

$dt = 2\theta\, d\theta \qquad d\theta = \dfrac{dt}{2\theta}$

$$= \int_{\frac{\pi}{2}}^{\pi} \theta^3 \cos(t)\, \frac{dt}{2\theta}$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \theta^2 \cos(t)\, dt$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} t\cos(t)\, dt = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} u\, dv$$

$u = t \qquad v = \sin(t)$

$du = dt \qquad dv = \cos(t)\, dt$

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} u\, dv = \frac{1}{2} uv\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} v\, du$$

$$= \frac{1}{2}\left(t\sin(t)\right)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \sin(t)\, dt$$

$$= \frac{1}{2}\left(t\sin(t)\right)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2}\left(-\cos(t)\right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{2}\left[t\sin(t) + \cos(t)\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{2}\left[\left[\pi(0) + -1\right] - \left[\frac{\pi}{2}(1) + 0\right]\right]$$

$$= \underline{-\frac{1}{2} - \frac{\pi}{4}}$$

Ex

$$\int \tan^{-1}(4t)\,dt = \int u\,dv$$

$u = \tan^{-1}(4t)$ $\quad$ $v = t$

$du = \dfrac{1}{1+(4t)^2}\cdot 4\,dt$ $\quad$ $dv = dt$

$$\int u\,dv = uv - \int v\,du$$

$$= \tan^{-1}(4t)\cdot t - \int t\cdot \frac{4}{1+(4t)^2}\,dt$$

$u = 4t$

$du = 4\,dt$

$dt = \dfrac{du}{4}$

$$= \quad '' \quad - \int \frac{4t}{1+(4t)^2}\,dt$$

$$= \tan^{-1}(4t)\cdot t - \int \frac{u}{1+u^2}\,\frac{du}{4}$$

$v = 1+u^2$

$dv = 2u\,du$

$du = \dfrac{dv}{2u}$

$$= t\cdot \tan^{-1}(4t) - \int \frac{u}{v}\,\frac{dv}{2u}\cdot\frac{1}{4}$$

$$= t\tan^{-1}(4t) - \frac{1}{8}\int \frac{dv}{v}$$

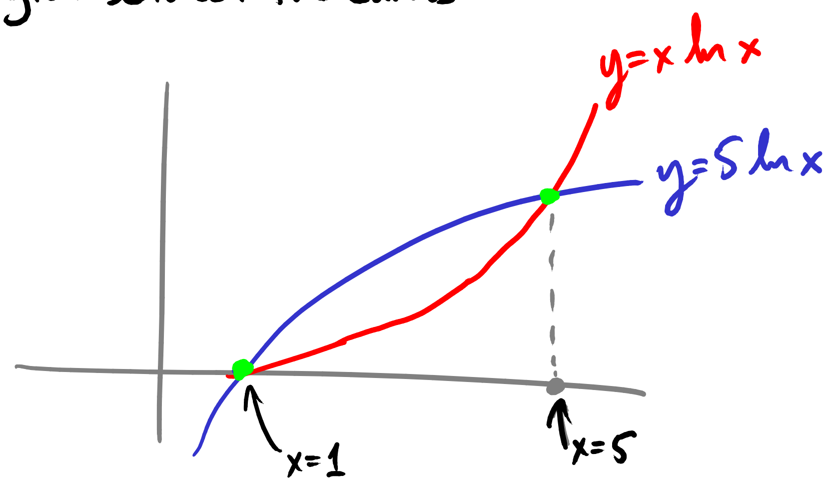
$$= t\tan^{-1}(4t) - \frac{1}{8}\ln|v|$$

$|v| = |1+u^2| = |1+16t^2|$

$$= t\tan^{-1}(4t) - \frac{1}{8}\ln(1+16t^2)$$

Ex Find the area of the region between the curves

$$y = 5\ln x$$
$$y = x\ln x$$



Intersections

$5\ln x = x\ln x$

$x = 5$ or $\ln x = 0$

$x = 1$

$\ln(1) = 0$

$$\int_1^5 (5\ln x - x\ln x)\,dx$$

$$= \int_1^5 (5-x)\ln x\,dx$$

Int by parts: $u = \ln x \qquad v = 5x - \frac{1}{2}x^2$

$du = \frac{1}{x}\,dx \qquad dv = (5-x)\,dx$

$$\int_1^5 u\,dv = uv\Big|_1^5 - \int_1^5 v\,du$$

$$= (\ln x)\left(5x - \tfrac{1}{2}x^2\right)\Big|_1^5 - \underbrace{\int_1^5 \left(5x - \tfrac{1}{2}x^2\right)\frac{1}{x}\,dx}_{}$$

$$= -\int_1^5 \left(5 - \tfrac{1}{2}x\right)dx$$

...

$$= \underline{\underline{\frac{25}{2}\ln 5 - 14}}$$

1 more substitution:

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\, dr$$

$u = 4 + r^2$

$du = 2r\, dr$

$dr = \frac{du}{2r}$

$$= \int_4^5 \frac{r^3}{\sqrt{u}} \frac{du}{2r}$$

$$= \int_4^5 \frac{1}{2} \frac{r^2}{\sqrt{u}}\, du$$

$r^2 = u - 4$

$$= \int_4^5 \frac{1}{2} \frac{u-4}{\sqrt{u}}\, du$$

$$= \int_4^5 \frac{1}{2} u^{1/2} - 2u^{-1/2}\, du$$

$= \dots$

$$= \underline{\underline{-\frac{7}{3}\sqrt{5} + \frac{16}{3}}}$$

Lecture 15 | 22 Feb 2010

[My web page: www.ma.utexas.edu/users/neitzke]

Exam WEL 1.316 7–9pm

EXAM REVIEW:

Int. by parts:

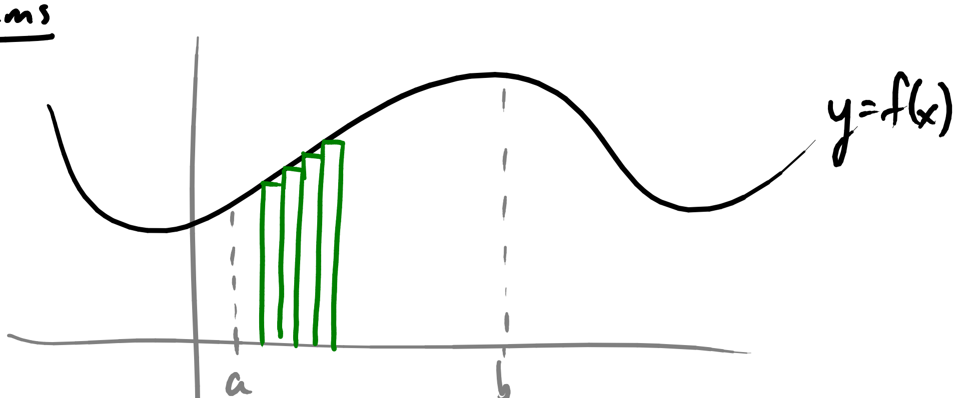
$$\int_0^1 16x^2 e^{2x}\,dx = \int_0^1 u\,dv$$

$$u = 16x^2 \qquad v = \tfrac{1}{2}e^{2x}$$
$$du = 32x\,dx \qquad dv = e^{2x}\,dx$$

$$\int_0^1 u\,dv = uv\Big|_0^1 - \int_0^1 v\,du$$
$$= 8x^2 e^{2x}\Big|_0^1 - \int_0^1 \tfrac{1}{2}e^{2x}\,32x\,dx$$

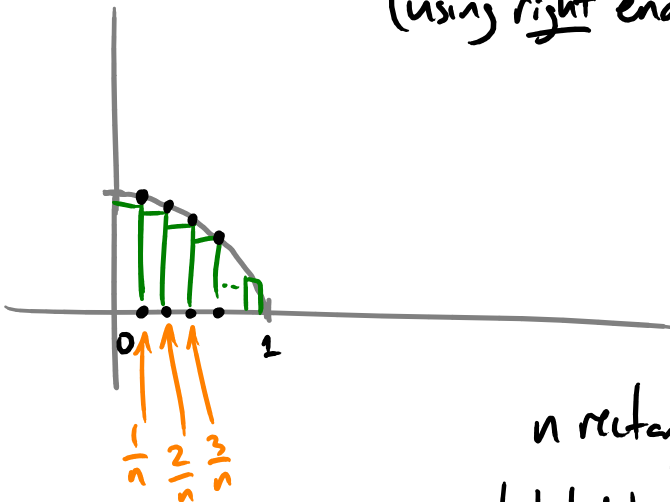
then do $\int$ by parts again...

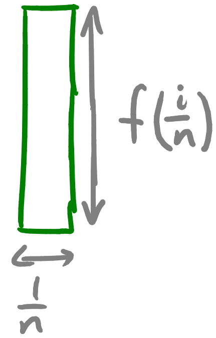
Riemann sums



Write the area as sum of areas of n little rectangles, then take $\lim_{n\to\infty}$ of the sum.

Ex Write $\int_0^1 (1-x^3)\,dx$ as a limit of Riemann sums (using right endpoints)





n rectangles

labeled by $i = 1, \dots, n$

$$\text{area} = \frac{1}{n} f\left(\frac{i}{n}\right) = \frac{1}{n}\left(1 - \left(\frac{i}{n}\right)^3\right)$$

So the Riemann sum is

$$A = \sum_{i=1}^{n} \frac{1}{n}\left(1 - \left(\frac{i}{n}\right)^3\right)$$

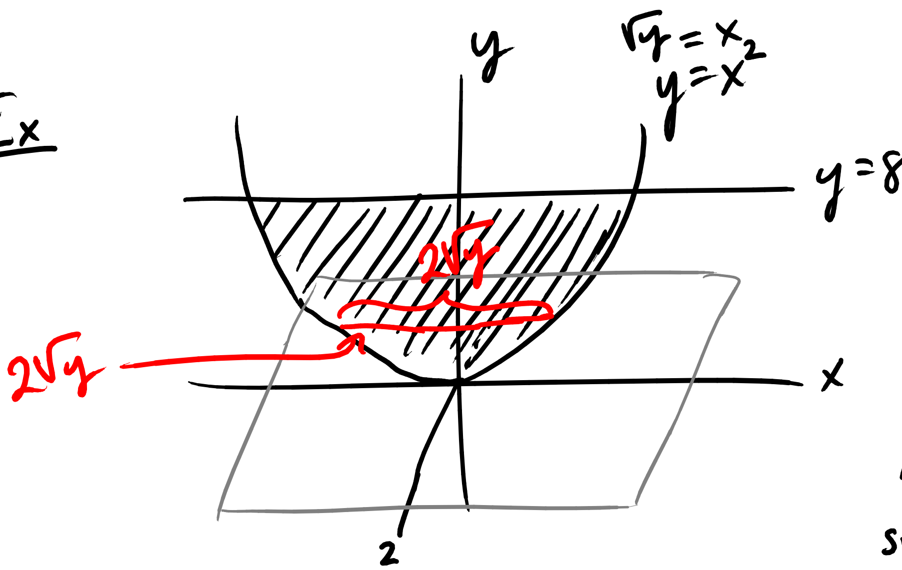
$$\left[\frac{i}{n} = a + i\Delta x\right]$$

Ex

$$\int \frac{\sin 2x}{\sin x}\,dx = ?$$

Use $\sin 2x = 2 \sin x \cos x$

Ex

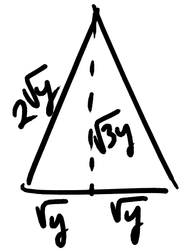


$$V = \int_0^8 dy\, A(y)$$

At fixed y, equilateral $\triangle$

side length $= 2\sqrt{y}$

$A(y) =$



$\frac{1}{2}bh = \frac{1}{2}(2\sqrt{y})(\sqrt{3y})$

$= \sqrt{3} \times y$

$$V = \int_0^8 \sqrt{3} \times y \, dy = \underline{\underline{32\sqrt{3}}}$$

FTC 1:

Ex If $F(x) = \int_5^x 3\sqrt{1+t^2}\, dt$

what is $F'(x)$?

$$F'(x) = \frac{d}{dx} \int_5^x 3\sqrt{1+t^2}\, dt = \underline{\underline{3\sqrt{1+x^2}}}$$

Ex If $F(x) = \int_5^{x^2} 3\sqrt{1+t^2}\, dt$

what is $F'(x)$?

[from the chain rule: $\frac{d}{dx} x^2$]

$$F'(x) = \frac{d}{dx} \int_5^{x^2} 3\sqrt{1+t^2}\, dt = 2x \cdot 3\sqrt{1+x^4}$$

$$= \underline{\underline{6x\sqrt{1+x^4}}}$$

Warnings:

$$\int \frac{1}{t^{1/3}}\,dt \neq \ln\left(t^{1/3}\right)$$

$$\int t^{-1/3}\,dt = \frac{3}{2}t^{2/3} + C$$

$$\int \frac{1}{t}\,dt = \ln|t|$$

$$\ln(a) + \ln(b) = \ln(ab)$$

$$-\ln(a) = \ln\left(\frac{1}{a}\right)$$

$$b\ln(a) = \ln\left(a^b\right)$$

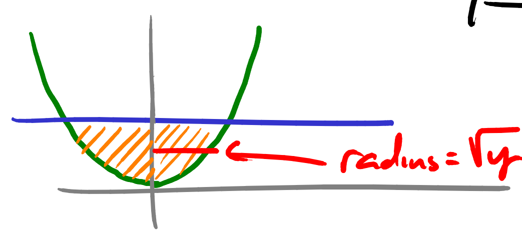
e.g.

$$\begin{aligned}\ln(9) - 2\ln(6) &= \ln(9) - \ln(36)\\ &= \ln\left(\frac{9}{36}\right)\\ &= \ln\left(\frac{1}{4}\right)\\ &= -\ln 4 = -2\ln 2\end{aligned}$$

Volumes of revolution

Calc the volume obt. by rotating the region bounded by

$y=x^2$, $y=1$ around the y-axis.



Cross-sections are circles area $A(y) = \pi r^2$

$= \pi(\sqrt{y})^2$

$= \pi y$

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi y\,dy = \underline{\underline{\frac{\pi}{2}}}$$

Ex $I = \int \frac{1}{2} \sin(\ln x)\,dx = \ ?$

Int by parts: $u = \sin(\ln x)$ . . .

. . $dv = dx$

$$\int e^x \sin x\,dx = (\text{stuff}) - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = (\text{stuff})$$

$$\int e^x \sin x\,dx = \frac{1}{2}(\text{stuff})$$

Lecture 16

24 Feb 2010

(More) trigonometric integrals (Ch 8.2)

$$\int \sin^5\theta \cos\theta \, d\theta$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= \int u^5 \, du$$

Similarly for $\int \sin^a\theta \cos\theta \, d\theta$

or for $\int \cos^b\theta \sin\theta \, d\theta$

But what about e.g. $\int \sin^3\theta \, d\theta$?

Ex $\int \sin^3\theta \, d\theta = \int \sin^2\theta \, (\sin\theta \, d\theta)$

Use $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

So $\int = \int (1-\cos^2\theta)(\sin\theta \, d\theta)$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$-du = \sin\theta \, d\theta$

$$= \int (1-u^2)(-du)$$

$$= \int (u^2 - 1) \, du = \frac{u^3}{3} - u + C = \underline{\underline{\frac{1}{3}\cos^3\theta - \cos\theta + C}}$$

Ex $\int \sin^5\theta \cos^2\theta \, d\theta$

$$= \int \sin^4\theta \cos^2\theta \, (\sin\theta \, d\theta)$$

Want $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$-du = \sin\theta \, d\theta$

$$= \int (\sin^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$$

$$= \int (1-\cos^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$$

$$= \int (1-u^2)^2 u^2 \, (-du)$$

$$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7}$$

$$= -\frac{\cos^3\theta}{3} + \frac{2\cos^5\theta}{5} - \frac{\cos^7\theta}{7}$$

General rule for $\int \sin^a\theta \cos^b\theta \, d\theta$:

If a odd, then pick off one of sines, write $(\sin\theta \, d\theta)$,
use $\sin^2\theta = 1-\cos^2\theta$ to eliminate the rest of the sines,
use $u = \cos\theta$.

If b odd, then pick off a cosine, write $(\cos\theta \, d\theta)$,
use $\cos^2\theta = 1-\sin^2\theta$ to elim. rest of cosines,
use $u = \sin\theta$.

What about <u>even</u> powers?

Half-angle formulas:

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$
$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

Ex
$$\int \sin^2\theta \, d\theta$$
$$= \int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$
$$= \int \frac{1}{2} - \frac{1}{2}\cos 2\theta \quad d\theta$$
$$= \frac{\theta}{2} - \frac{1}{2}\left(\frac{1}{2}\sin 2\theta\right) + C = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C$$

Ex
$$\int \cos^4\theta \, d\theta$$
$$= \int (\cos^2\theta)^2 \, d\theta$$
$$= \int \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$
$$= \frac{1}{4}\int \left(1+2\cos 2\theta + \cos^2 2\theta\right) d\theta$$
$$= \frac{1}{4}\int \left(1+2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right) d\theta$$
$$= \frac{1}{4}\int \left(\frac{3}{2}+2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$
$$= \frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C$$

Ex $\int \tan^6 x \sec^4 x \, dx = ?$

Similar rules to what we used for sin, cos above:

$$\int = \int \tan^6 x \sec^2 x \, (\sec^2 x \, dx)$$

Want $u = \tan x$
$du = \sec^2 x \, dx$

Use $\sec^2 x = 1 + \tan^2 x$

$$= \int \tan^6 x \, (1 + \tan^2 x) \, (\sec^2 x \, dx)$$

$$= \int u^6 (1 + u^2) \, du$$

$$= \int (u^6 + u^8) \, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \underline{\underline{\frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C}}$$

Same strategy whenever <u>sec x</u> appears to an <u>even</u> power.

Ex $\int_0^{\pi/4} \tan^3 x \sec^5 x \, dx$

$$= \int_0^{\pi/4} \tan^2 x \sec^4 x \, (\tan x \sec x) \, dx$$

Want $u = \sec x$
$du = \sec x \tan x \, dx$

use $\tan^2 x = \sec^2 x - 1$

$$\int_0^{\pi/4} (\sec^2 x - 1) \sec^4 x \, (\tan x \sec x) \, dx$$

$$= \int_1^{\sqrt{2}} (u^2 - 1) u^4 \, du$$

$$= \cdots \qquad = \frac{2}{35}(1 + 6\sqrt{2})$$

Same strategy works for $\int \tan^a x \sec^b x \, dx$

whenever a is odd (and $b \geq 1$)

Handy facts:

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Ex $\displaystyle\int \tan^3 x\, dx$

$$= \int \tan x \cdot \tan^2 x\, dx$$

$$= \int \tan x \left(\sec^2 x - 1\right) dx$$

$$= \int \tan x \sec^2 x\, dx - \int \tan x\, dx$$

$$\left[\begin{array}{l} u = \tan x \\ du = \sec^2 x \end{array} \Rightarrow \int u\, du = \frac{1}{2}u^2 \right]$$

$$= \underline{\underline{\frac{1}{2}\left(\tan^2 x\right) - \ln|\sec x| + C}}$$

Ex $\displaystyle\int \sin 4x \cos 7x\, dx$

Use product-to-sum identities:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$
$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$= \int \frac{1}{2}\left(\sin(-3x) + \sin(11x)\right) dx$$

$$= \underline{\underline{\frac{1}{2}\left(\frac{1}{3}\cos(3x) - \frac{1}{11}\cos(11x)\right) + C}}$$

Lecture 17 | 26 Feb 2010

Last time: trigonometric $\int$s

Like $\int \sin^a\theta \cos^b\theta \, d\theta$

$\int \sec^a\theta \tan^b\theta \, d\theta$

One more example:

Ex $I = \int \sec^3 x \, dx$

Int. by parts: $u = \sec x$ $\quad v = \tan x$

$du = \sec x \tan x \, dx$ $\quad dv = \sec^2 x \, dx$

$$I = \int u \, dv = uv - \int v \, du$$

$$= \sec x \tan x - \int \tan x \sec x \tan x \, dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

Use $\tan^2 x = \sec^2 x - 1$

$$I = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$I = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx$$

$$I = \sec x \tan x + \ln|\sec x + \tan x| + C - I$$

$$2I = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\underline{I = \tfrac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C}$$

Next method:

Trigonometric Substitution (Ch 8.3)

Ex $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = ?$

$\left[-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right]$

A clever substitution: $x = 3\sin\theta$ [since $-3 \leq x \leq 3$]

$dx = 3\cos\theta\, d\theta$

$$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta}$$

$$= 3\sqrt{\cos^2\theta}$$

(since $\cos\theta > 0$) $= 3\cos\theta$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{(3\sin\theta)^2}\cdot 3\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta\, d\theta$$

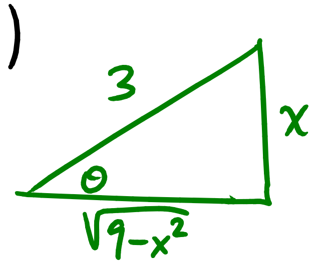
$$= \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C$$

and remember that $x = 3\sin\theta$ $\quad\theta = \sin^{-1}\left(\frac{x}{3}\right)$

To get $\cot\theta$ in terms of x:

$\frac{x}{3} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$



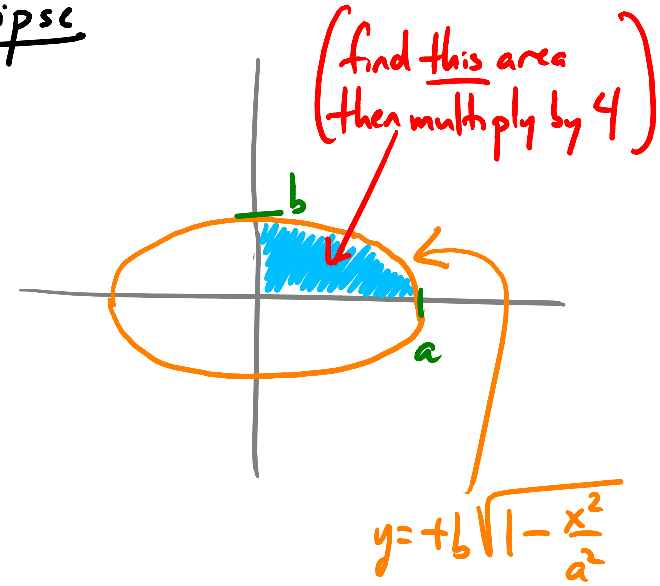
$\cot\theta = \frac{adj}{opp} = \frac{\sqrt{9-x^2}}{x}$ $\tan\theta = \frac{opp}{adj}$

So finally, the integral is

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Ex Find the area enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



solve for y: $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$A=4\int_0^a dx\, b\sqrt{1-\frac{x^2}{a^2}}$$

$$=4b\int_0^a dx\sqrt{1-\frac{x^2}{a^2}}$$

Want this to become $\sqrt{1-\sin^2\theta}$

So, subst. $\frac{x^2}{a^2}=\sin^2\theta$

ie $x=a\sin\theta$

$dx=a\cos\theta\, d\theta$

Change limits: $x=0$ is $\theta=0$

$x=a$ is $\theta=\frac{\pi}{2}$

[because $\sin\theta=1$]

$$=4b\int_0^{\frac{\pi}{2}} a\cos\theta\, d\theta\cdot\sqrt{1-\sin^2\theta}$$

$$=4ba\int_0^{\frac{\pi}{2}}\cos\theta\, d\theta\sqrt{\cos^2\theta}$$

$$=4ba\int_0^{\frac{\pi}{2}}\cos\theta\, d\theta\,\cos\theta$$

$$=4ba\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta$$

$$= 4ba \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta \qquad \left(\tfrac{1}{2}\text{-angle identity}\right)$$

$$= 4ba \int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2\theta \; d\theta$$

$$= \cdots$$

$$= \underline{\underline{ab\pi}}$$

Ex

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

Here use the identity $\tan^2+1=\sec^2$:

Substitute

$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\, d\theta$$

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4}$$
$$= 2\sqrt{\tan^2\theta+1}$$
$$= 2\sqrt{\sec^2\theta}$$
$$= 2\sec\theta$$

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx = \int \frac{1}{4\tan^2\theta\cdot 2\sec\theta}\cdot 2\sec^2\theta\, d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\cos\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u=\sin\theta$:
$du=\cos\theta\, d\theta$

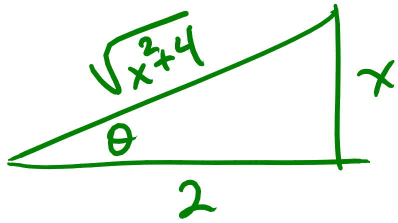
$$= \frac{1}{4}\int \frac{du}{u^2} = -\frac{1}{4}\left(\frac{1}{u}\right)+C$$

Subst. back to original variable:

$u = \sin\theta$
$x = 2\tan\theta$
$\frac{x}{2} = \tan\theta$

What's u in terms of x?

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{x}{2}$$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+4}}$$

so $\quad u = \dfrac{x}{\sqrt{x^2+4}}$

and the integral is $\quad -\dfrac{1}{4u} = \underline{\underline{-\dfrac{1}{4}\dfrac{\sqrt{x^2+4}}{x} + C}}$

Table: $\sqrt{a^2-x^2}$ use $x = a\sin\theta$, $1-\sin^2\theta = \cos^2\theta$

$\sqrt{a^2+x^2}$ use $x = a\tan\theta$, $1+\tan^2\theta = \sec^2\theta$

$\sqrt{x^2-a^2}$ use $x = a\sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$

Lecture 18

1 March 2010

Notes are at http://www.ma.utexas.edu/users/neitzke

A difficult HW pb:

$$\int_0^a \sqrt{1+\sin\theta}\, d\theta$$

Could try: $\displaystyle\int_0^a \sqrt{1+\sin\theta}\,\frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\,d\theta = \int_0^a \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}}\,d\theta = \int_0^a \frac{\cos\theta\, d\theta}{\sqrt{1-\sin\theta}}$

$$u = 1-\sin\theta \qquad du = -\cos\theta\, d\theta \ldots$$

But this sometimes doesn't work because $1-\sin\theta$ could be zero somewhere in $0<\theta<a$. Then you'd get wrong answer!

Set $\theta = 2x$: $\sin\theta = \sin 2x = 2\sin x\cos x$

And use $1 = \cos^2 x + \sin^2 x$. Then $\sqrt{1+\sin\theta} = \sqrt{\cos^2 x + \sin^2 x + 2\sin x\cos x}$

$(x = \frac{\theta}{2})$

$$= \sqrt{(\sin x + \cos x)^2}$$

$$= |\sin x + \cos x|$$

Last time: trig substitution

If you see $\sqrt{a^2-x^2}$ try $x = a\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad$ i.e. $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$\sqrt{a^2+x^2}$ try $x = a\tan\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\quad \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$\sqrt{x^2-a^2}$ try $x = a\sec\theta$ $\quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ $\quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$

Ex $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx$ $\qquad$ try $x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$$= \int_0^{\pi/3} \frac{(4\sin\theta)^3}{\sqrt{16-(4\sin\theta)^2}} \cdot 4\cos\theta\,d\theta$$

limits: $x=0$ is $\sin\theta = 0$

ie $\theta = 0$

$x = 2\sqrt{3}$ is $\sin\theta = \frac{\sqrt{3}}{2}$

ie $\theta = \frac{\pi}{3}$

$$= \int_0^{\pi/3} \frac{4^3\sin^3\theta}{\sqrt{16(1-\sin^2\theta)}} \cdot 4\cos\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\sqrt{\cos^2\theta}} \cdot 4\cos\theta\,d\theta$$

$$= 4^3\int_0^{\pi/3} \sin^3\theta \cdot \frac{\cos\theta}{\cos\theta}\,d\theta$$

$$= 4^3\int_0^{\pi/3} \sin^3\theta\,d\theta$$

$$= 4^3\int_0^{\pi/3} \sin^2\theta \cdot (\sin\theta\,d\theta)$$

Want $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$$= 4^3\int_0^{\pi/3} (1-\cos^2\theta)(\sin\theta\,d\theta) = 4^3\int_1^{1/2} (1-u^2)(-du) = \cdots = \frac{40}{3}$$

$\boxed{\sin^2\theta + \cos^2\theta = 1}$

Ex $\displaystyle\int \frac{dx}{\sqrt{x^2+8x+25}}$

Want to relate this to s.t. like $\displaystyle\int \frac{1}{\sqrt{u^2+a^2}}$

Complete the square: $u = x + c$ for some constant c

$$u^2 = x^2 + 2c\cdot x + c^2$$

take $c=4$, ie $u = x+4$, then $u^2 = x^2+8x+16$

So $u^2+9 = x^2+8x+25$ $\qquad du = dx$

Then $\displaystyle\int \frac{dx}{\sqrt{x^2+8x+25}} = \int \frac{du}{\sqrt{u^2+9}}$

So we can substitute $u = 3\tan\theta$ $\qquad du = 3\sec^2\theta \, d\theta$

Then
$$\int = \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9\tan^2\theta + 9}}$$

$$= \int \frac{3\sec^2\theta}{3\sqrt{\tan^2\theta+1}} \, d\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$\left(= \int \frac{3\sec^2\theta}{3\sec\theta} \, d\theta \right)$$

$$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} \, d\theta = \int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

Write it in terms of x: use $u = 3\tan\theta \to \tan\theta = \frac{u}{3} = \frac{x+4}{3}$

To get $\sec\theta$ in terms of x, could draw $\triangle$ and use SOH CAH TOA.

But we know $\sec\theta = \frac{1}{3}\sqrt{x^2+8x+25}$!

So altogether $\displaystyle\int \frac{dx}{\sqrt{x^2+8x+25}} = \underline{\underline{\ln\left|\frac{1}{3}\sqrt{x^2+8x+25} + \frac{1}{3}(x+4)\right|}}$

Ex $\displaystyle\int x\sqrt{1-x^4}\,dx$

$u = x^2 \qquad du = 2x\,dx$

$u^2 = x^4 \qquad \frac{1}{2}du = x\,dx$

$$= \int \sqrt{1-u^2}\,(x\,dx)$$

$$= \frac{1}{2}\int \sqrt{1-u^2}\,du$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$x^2 = u$

$$= \frac{1}{2}\int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

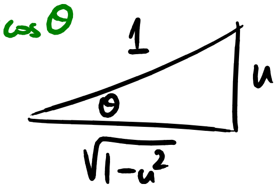
$$= \frac{1}{2}\int \cos^2\theta\,d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$



$\sin\theta = u$

$\cos\theta = \sqrt{1-u^2}$

$$= \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + C$$

$$= \frac{1}{4}\sin^{-1}(u) + \frac{1}{4}u\sqrt{1-u^2} + C$$

$$= \underline{\underline{\frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1-x^4} + C}}$$

Hwks: "REVIEW 1" is not another HW assignment
(just for you to review for the final!)

Partial fractions (Sec 8.4)

How to integrate complicated rational functions $\frac{P(x)}{Q(x)}$
[P, Q polynomials]

Ex. $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

Factor the denominator:

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(2x-1)(x+2)$$

Then set $\displaystyle \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$

To find A, B, C: multiply both sides by the denominator $x(2x-1)(x+2)$

$$x^2+2x-1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1)$$
$$= A(2x^2+3x-2) + B(x^2+2x) + C(2x^2-x)$$
$$= (2A+B+2C)x^2 + (3A+2B-C)x - 2A\,(1)$$

Equate the coefficients:

$$1 = 2A+B+2C$$
$$2 = 3A+2B-C$$
$$-1 = -2A$$

Solve these eq: $A=\frac{1}{2}$

$B=\frac{1}{5}$

$C=-\frac{1}{10}$

$1=1+B+2C$

$2=\frac{3}{2}+2B-C$

$B+2C=0$

$B=-2C$

$2=\frac{3}{2}-4C-C$

$\frac{1}{2}=-5C$

$C=-\frac{1}{10}$

So $$\frac{x^2+2x-1}{2x^3+3x^2-2x}=\frac{1}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2}$$

$$\int\frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx=\int\left(\frac{1}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2}\right)dx$$

$$=\underline{\underline{\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+K}}$$

What if the denominator doesn't factor completely (into linear factors)?

Ex $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$

Factor: $x^3+4x = x(x^2+4)$

Write $\displaystyle\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

To find A, B, C: mult. both sides by $x(x^2+4)$

$$\begin{aligned} 2x^2-x+4 &= A(x^2+4)+(Bx+C)(x) \\ &= Ax^2+4A+Bx^2+Cx \\ &= (A+B)x^2+Cx+4A \end{aligned}$$

Equate coeff: $\left.\begin{aligned} 2&=A+B \\ -1&=C \\ 4&=4A \end{aligned}\right\} \longrightarrow \begin{aligned} A&=1 \\ B&=1 \\ C&=-1 \end{aligned}$

So $\displaystyle\int = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx$

$$= \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

$\ln|x|$

use $u=x^2+4$ get $\frac{1}{2}\ln(x^2+4)$

use $u=\frac{x}{2}$, get $-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

$$= \underline{\ln|x| + \tfrac{1}{2}\ln(x^2+4) - \tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right) + K}$$

What if the degree of the numerator $\geq$ the degree of the denominator?

Ex $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

Divide first:

$$\begin{array}{r|l} & x+1 \\ \hline x^2-x-6 & x^3+0x^2-4x-10 \\ & x^3-x^2-6x \\ \hline & \qquad x^2+2x-10 \\ & \qquad x^2-x-6 \\ \hline & \qquad\quad 3x-4 \end{array}$$

So $\displaystyle\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{x^2-x-6}$

So $\displaystyle\int \frac{x^3-4x-10}{x^2-x-6} = \int x+1+\frac{3x-4}{x^2-x-6}\,dx$

Factor: $x^2-x-6=(x-3)(x+2)$

$$\frac{3x-4}{x^2-x-6} = \frac{A}{x-3}+\frac{B}{x+2}$$

$$3x-4 = A(x+2)+B(x-3)$$

$$3x-4 = (A+B)x+(2A-3B)$$

$$\left.\begin{array}{r} 3 = A+B \\ -4 = 2A-3B \end{array}\right] \Rightarrow A=1, B=2$$

So $\displaystyle\int = \int_0^1 x+1+\frac{1}{x-3}+\frac{2}{x+2}\,dx = \cdots = \underline{\underline{\frac{3}{2}+\ln\frac{3}{2}}}$

What if some factor appears <u>more than once</u> in the denom?

<u>Ex</u> $\displaystyle\int \frac{1}{x^3+2x^2+x}\,dx$

Factor: $x^3+2x^2+x = x(x^2+2x+1)$
$= x(x+1)^2$

Write $$\frac{1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$!

Mult. by $x(x+1)^2$ both sides:

$$\begin{aligned} 1 &= A(x+1)^2 + Bx(x+1) + Cx \\ &= A(x^2+2x+1) + B(x^2+x) + Cx \\ &= (A+B)x^2 + (2A+B+C)x + A \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} A+B&=0 \\ 2A+B+C&=0 \\ A&=1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A&=1 \\ B&=-1 \\ C&=-1 \end{aligned}$$

Lecture 20

5 Mar 2010

Last time: <u>partial fractions</u>

$$\int \frac{P(x)}{Q(x)}\,dx \qquad \text{e.g.} \qquad \int \frac{x^4+4x^3+2x-17}{x^3+2x^2+x}\,dx$$

first do <u>polynomial long division</u> (if deg (numerator) $\geq$ deg (denom))

then <u>factor the denominator</u> and use that to split up the fraction

Can even use this on simple-looking things:

e.g. $$\int \frac{1}{x(x+1)}\,dx$$

Don't have to do long division here: just write

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Mult. by $x(x+1)$ both sides:

$$1 = A(x+1) + Bx$$
$$1 = (A+B)x + A \Longrightarrow \left[\begin{array}{l} A=1 \\ A+B=0 \end{array}\right] \Rightarrow \left[\begin{array}{l} A=1 \\ B=-1 \end{array}\right]$$

<u>so</u> $$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

So $$\int \frac{dx}{x(x+1)} = \int dx\left(\frac{1}{x} - \frac{1}{x+1}\right) = \underline{\underline{\ln|x| - \ln|x+1| + C}}$$

If we had instead $\int \frac{1}{x(x^2+1)}$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Strategy For Integration (Ch 8.5)

1) Basic integration formulas

$\int x^n\,dx = \ldots$

$\int \sin x\,dx = \ldots$

$\int \frac{1}{x}\,dx = \ldots$

(table p. 520 of text)

2) Simplify the integrand (using either algebra or trig identities)

$$\int \sqrt{x}(1+\sqrt{x})\,dx = \int (\sqrt{x}+x)\,dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta\,d\theta = \int \sin\theta\cos\theta\,d\theta = \frac{1}{2}\int \sin 2\theta\,d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

3) "Easy" substitutions

$$\int \frac{x}{x^2-1}\,dx \qquad \begin{array}{l} u = x^2-1 \\ du = 2x\,dx \end{array} \quad \ldots$$

4) Classify:

a) Trig $[\sin^a x\cos^b x,\ \tan^a x\sec^b x,\ \cot^a x\csc^b x]$ — use rules from last week

b) Rational function — partial fractions

c) Product of 2 distinct pieces — int. by parts

d) Radicals — $\sqrt{\pm x^2 \pm a^2}$ — trig sub

$\sqrt[n]{ax+b}$ — $u = \sqrt[n]{ax+b}$ sub

5) Try again:

a) look for a clever substitution

b) $\int$ by parts [even if you have to take $dv = dx$, $u =$ the whole integrand

e.g. $\int \tan^{-1} x \, dx$ can be done this way]

c) algebraic manip. [e.g. $\int \frac{dx}{1-\cos x}$: multiply top & bottom by $1+\cos x$]

d) try to relate it to one you've done before

e) combine several methods...

Ex $\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$

[or: use $u=\frac{x}{3}$ and the fact $\int \frac{1}{1+u^2}\,du = \tan^{-1}(u)$]

Ex $\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \tan^3 x \sec^3 x\,dx$

odd # of powers of tangent $\Rightarrow \int \tan^2 x \sec^2 x \,(\tan x \sec x\,dx)$

$u = \sec x$
$du = \tan x \sec x$

use $\tan^2 x = \sec^2 x - 1$

...

Ex $\int e^{\sqrt{x}}\,dx$ Use $u = \sqrt{x}$ $x = u^2$
$dx = 2u\,du$

$\int = \int e^u (2u\,du)$ Use $\int$ by parts...

Ex $\int \frac{x^5+1}{x^3-3x^2-10x}\,dx$ Partial fractions

Ex $\int \frac{dx}{x\sqrt{\ln x}}$ $u = \ln x$ $\left(du = \frac{dx}{x}\right)$

$= \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$

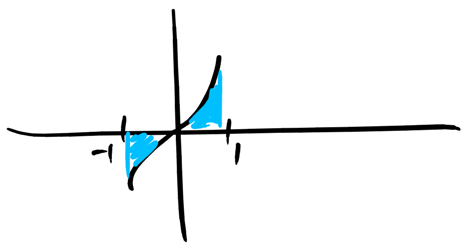
Ex $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$ mult. top, bottom by $\sqrt{1-x}$

$$\longrightarrow \int \frac{1-x}{\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

trig sub: $x = \sin\theta$ (or just $\int = \sin^{-1}x$)

u-sub: $u = 1-x^2$

Ex $\displaystyle\int_{-1}^{1} x^4 \sin x \, dx$ One way: $\int$ by parts (4 times)



Or: use the fact that

$$\int_{-a}^{a} (\text{odd function})\,dx = \underline{\underline{0}}$$

Ex $\displaystyle\int (x+\sin x)^2\,dx$

Multiply out: $\displaystyle\int x^2 + 2x\sin x + \sin^2 x \;dx$

easy by parts $\frac{1}{2}$-angle identity

Ex $$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{dx}{\left(\sqrt{1-x^2}\right)^3}$$

$x = \sin\theta$
$dx = \cos\theta\,d\theta$

$$= \int \sec^2\theta\,d\theta = \underline{\underline{\tan\theta + C}}$$

Lecture 21

8 Mar 2010

Indeterminate forms and L'Hospital's Rule (Ch 7.8)

$$F(x) = \frac{x^2+2}{x^2-4}$$

What's $\lim_{x\to 1} F(x)$? Both num. and denom. stay finite (and nonzero):

$$\lim_{x\to 1} F(x) = \frac{1^2+2}{1^2-4} = \frac{3}{-3} = \underline{\underline{-1}}$$

$\lim_{x\to 2} F(x)$? Numerator $\to 2^2+2 = 6$

Denominator $\to 2^2-4 = 0$

So we're getting $\frac{6}{0}$ here...

If $x\to 2$ from the <u>positive</u> direction, then we have $\frac{6}{\text{(very small positive)}}$

so $\lim_{x\to 2^+} F(x) = +\infty$

If $x \to 2$ from the <u>negative</u> direction we have $\frac{6}{\text{(very small negative)}}$

So $\lim_{x\to 2^-} F(x) = -\infty$

But what about more complicated cases where we get...

$\frac{0}{0}$, or $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞, ...

How do we deal with these "indeterminate forms"?

Sometimes just algebraic manipulations:

Ex $$\lim_{x\to\infty} \frac{x^2+3x+4}{3x^2-6x-7}$$

If we take the limits of num, denom separately, get $\frac{\infty}{\infty}$.

But we can divide both num, denom by x^2:

then get $$\lim_{x\to\infty} \frac{(x^2+3x+4)\cdot\left(\frac{1}{x^2}\right)}{(3x^2-6x-7)\cdot\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{1+\frac{3}{x}+\frac{4}{x^2}}{3-\frac{6}{x}-\frac{7}{x^2}}$$

Now take lim of num, denom: get

$$= \frac{\lim_{x\to\infty} 1+\frac{3}{x}+\frac{4}{x^2}}{\lim_{x\to\infty} 3-\frac{6}{x}-\frac{7}{x^2}} = \frac{1}{3}$$

Sometimes those kinds of methods don't help: use L'Hospital's Rule

L's rule: If $\left[\begin{array}{c}\lim_{x\to a} f(x) = \pm\infty \\ \text{and} \\ \lim_{x\to a} g(x) = \pm\infty\end{array}\right]$ OR $\left[\begin{array}{c}\lim_{x\to a} f(x) = 0 \\ \text{and} \\ \lim_{x\to a} g(x) = 0\end{array}\right]$

THEN $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Ex $\displaystyle\lim_{t\to 0} \frac{e^t-1}{t}$ Try $\dfrac{\lim_{t\to 0} e^t-1}{\lim_{t\to 0} t}$: that's $\dfrac{0}{0}$

so need L'H rule:

$$\lim_{t\to 0} \frac{e^t-1}{t} = \lim_{t\to 0} \frac{\frac{d}{dt}(e^t-1)}{\frac{d}{dt}(t)} = \lim_{t\to 0} \frac{e^t}{1} = \frac{1}{1} = \underline{\underline{1}}$$

Ex $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}$ $\dfrac{\lim_{x\to 0} \sin 4x}{\lim_{x\to 0} \tan 5x} = \dfrac{0}{0}$ so need L'H:

$$= \lim_{x\to 0} \frac{\frac{d}{dx}\sin 4x}{\frac{d}{dx}\tan 5x} = \lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x} = \frac{4\cdot 1}{5\cdot 1} = \underline{\underline{\frac{4}{5}}}$$

Ex $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$: $\dfrac{\infty}{\infty}$ so need L'H rule

$$= \lim_{x\to\infty} \frac{e^x}{2x} : \frac{\infty}{\infty} \quad \text{use L'H again}$$

$$= \lim_{x\to\infty} \frac{e^x}{2} : \frac{\infty}{2}, \text{ ie } \underline{\underline{\lim_{x\to\infty} \frac{e^x}{x^2} = \infty}}$$

Ex $\displaystyle\lim_{x\to\pi} \frac{\sin x}{1-\cos x} = \frac{0}{2} = \underline{\underline{0}}$ (No L'H rule here!)

To deal with $0 \cdot \infty$:

Try to rewrite it as $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Ex $\lim_{x\to 0^+} x \ln x$ $\quad\left[\begin{array}{l} \lim_{x\to 0^+} x = 0 \\ \lim_{x\to 0^+} \ln x = -\infty \end{array} \text{ so this is } 0\cdot(-\infty)\right]$

$$= \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \quad \left[\text{this is } \frac{\infty}{\infty} \text{ so can use L'H rule}\right]$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} (-x) = \underline{\underline{0}}$$

Ex $\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right)$ $\quad$ looks like $\infty \cdot 0$: rewrite it

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\left(\frac{1}{x}\right)} \quad \left[\to \frac{0}{0} \text{ so use L'H}\right]$$

$$= \lim_{x\to\infty} \frac{\left(-\pi/x^2\right)\left(\cos\frac{\pi}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty} \pi\cos\left(\frac{\pi}{x}\right) = \pi\cos(0) = \underline{\underline{\pi}}$$

For $\infty - \infty$:

again try to make it into $\frac{\infty}{\infty}$ or $\frac{0}{0}$

Ex $\lim_{x \to (\pi/2)^-} \sec x - \tan x \quad (\to \infty - \infty)$

$$= \lim_{x \to (\frac{\pi}{2})^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$

$$= \lim_{x \to (\frac{\pi}{2})^-} \frac{1 - \sin x}{\cos x} \to \frac{0}{0} \text{ so use L'H:}$$

$$= \lim_{x \to (\frac{\pi}{2})^-} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = \underline{\underline{0}}$$

Lecture 22

10 Mar 2010

Housekeeping:
- Exam 2 Tue Apr 6 7-9pm WEL 1.316
- HW 9 due Tue Mar 23 3am

Last time: Indeterminate forms and L'Hospital's rule

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$ we discussed how to do

But what about 1^∞, 0^0, ∞^0?

Basic strategy: take the log and evaluate its limit

$$\lim f(x) = e^{\lim \log f(x)}$$

Ex: $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx} \quad [\longrightarrow 1^\infty]$

$$= e^{\lim_{x\to\infty} \log\left(1+\frac{a}{x}\right)^{bx}}$$

and $\lim_{x\to\infty} \log\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} bx \log\left(1+\frac{a}{x}\right) \quad [\longrightarrow \infty\cdot 0]$

$$= \lim_{x\to\infty} \frac{\log\left(1+\frac{a}{x}\right)}{\left(\frac{1}{bx}\right)} \quad \left[\longrightarrow \frac{0}{0}\right]$$

use L'H:

$$= \lim_{x\to\infty} \frac{\left(-\frac{a}{x^2}\right)\cdot\left(\frac{1}{1+\frac{a}{x}}\right)}{\left(-\frac{1}{bx^2}\right)}$$

$$= \lim_{x\to\infty} \frac{ab}{1+\frac{a}{x}} = \frac{ab}{1} = ab$$

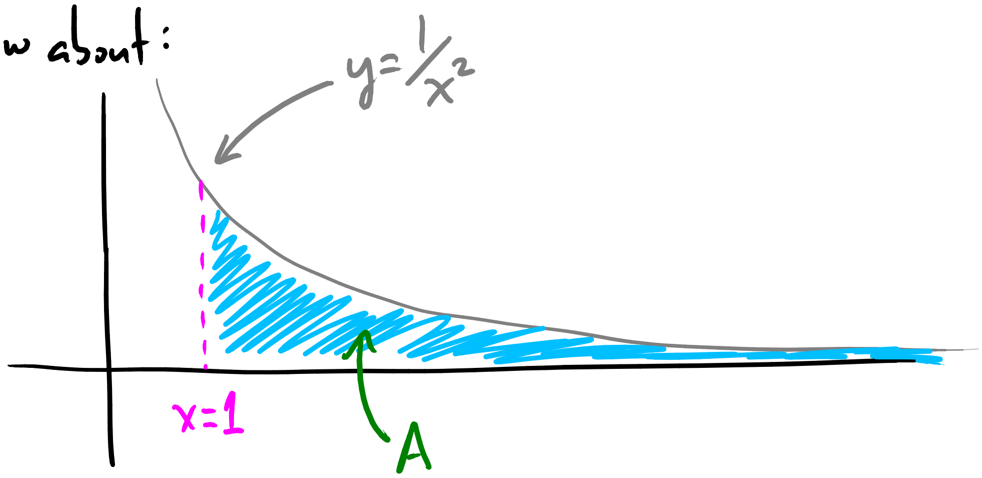
So $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \underline{\underline{e^{ab}}}$

Improper Integrals (Ch 8.8)

The $\int$ we did so far, $\int_a^b f(x)\,dx$ where $f(x)$ was well defined, finite for all $a \leq x \leq b$.

Those are called proper (definite) integrals.

How about:



the area of this infinite region?

$$A = \int_1^\infty \frac{1}{x^2}\,dx$$

This really means: define $A(t) = \int_1^t \frac{1}{x^2}\,dx$

then $A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx.$

So, let's calculate it:

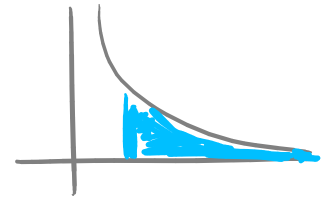
$$\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{x} \Big|_1^t \right) = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = \underline{\underline{1}}$$

$$\underline{\underline{A = 1}}$$

When the limit exists (like it did here), we call the improper integral <u>convergent</u>.

$$\left[\int_1^\infty \frac{1}{x^2}\,dx \quad \text{is } \underline{\text{convergent}}. \right]$$

How about $\displaystyle\int_1^\infty \frac{1}{x}\,dx$?



$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\left[\left(\ln x \Big|_1^t\right)\right]$$

$$= \lim_{t\to\infty} (\ln t)$$

This limit doesn't exist (goes to $+\infty$)

This improper integral is <u>not</u> convergent — it is <u>divergent</u>.

[If the limit is $+\infty$ or $-\infty$, <u>or</u> doesn't exist, call it <u>divergent</u>]

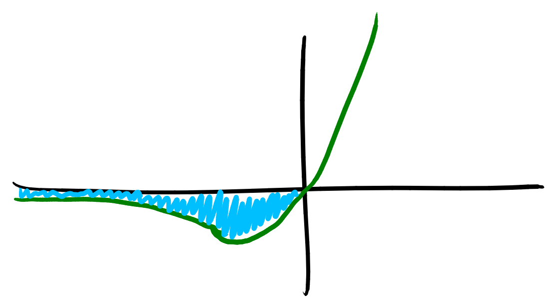
So far we saw: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is <u>convergent</u>

$\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is <u>divergent</u>

General rule:

$$\int^\infty \frac{1}{x^p}\,dx \text{ is } \begin{cases} \underline{\text{convergent}} \text{ for } p>1 \\ \underline{\text{divergent}} \text{ for } p\le 1 \end{cases}$$

<u>Ex</u> $\displaystyle\int_{-\infty}^0 x e^x\,dx$



$$= \lim_{t\to-\infty} \int_t^0 x e^x\,dx$$

(integrate by parts)

$$= \lim_{t\to-\infty}\left[-te^t - 1 + e^t\right]$$

$\infty \cdot 0$ $\qquad$ $\to 0$

use L'H rule for the 1st term: $\lim\limits_{t\to-\infty}(-te^{t}) = \lim\limits_{t\to-\infty} \dfrac{-t}{e^{-t}}$ $\quad \dfrac{\infty}{\infty}$

$$\overset{(L'H)}{=} \lim_{t\to-\infty} \frac{-1}{-e^{-t}}$$

$$= \lim_{t\to-\infty} \frac{1}{e^{-t}} = \frac{1}{\infty} = \underline{\underline{0}}$$

FALSE START:

$$= \lim_{t\to-\infty} \frac{e^{t}}{\left(-\frac{1}{t}\right)} \quad \frac{0}{0}$$

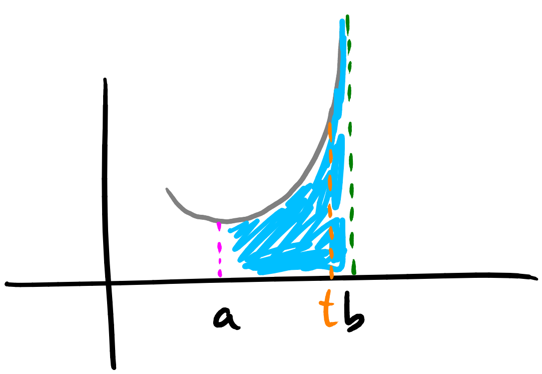
$$\overset{(L'H)}{=} \lim_{t\to-\infty} \frac{e^{t}}{\frac{1}{t^2}}$$

So altogether $\displaystyle\int_{-\infty}^{0} xe^{-x}\,dx = \underline{\underline{-1}}$ (<u>convergent</u>)

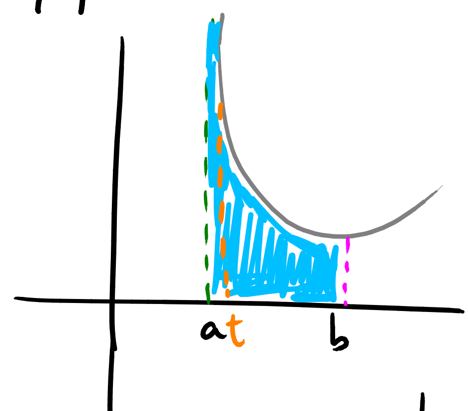
Another kind of improper integral:

$\displaystyle\int_a^b f(x)\,dx$ where $f(x)$ becomes infinite somewhere

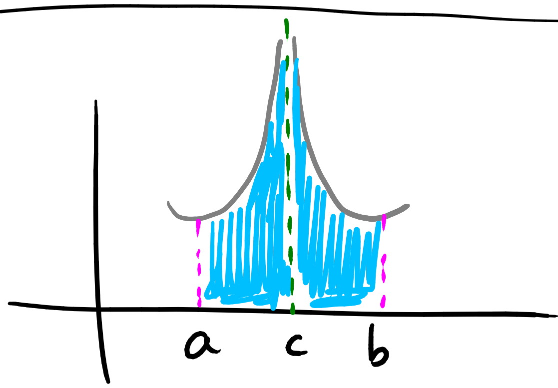
[= $f(x)$ has a vertical asymptote]



Here $\displaystyle\int_a^b f(x)\,dx$ <u>means</u> $\displaystyle\lim_{t\to b^-}\int_a^t f(x)\,dx$



Here $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$



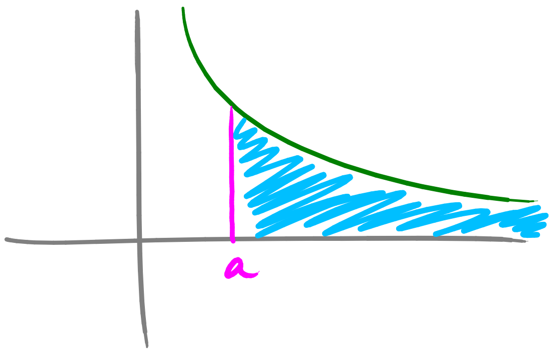
Here $\displaystyle\int_a^b f(x)\,dx$ <u>means</u>

$$\lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$$

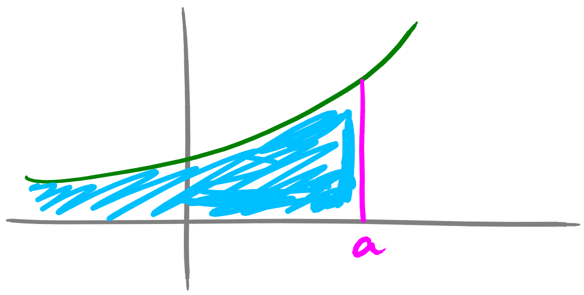
Lecture 23

12 Mar 2010

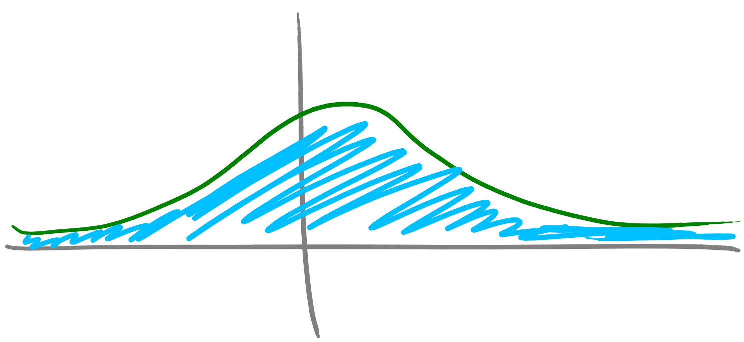
Last time: improper int.



$$\int_a^{\infty} f(x)\,dx$$



$$\int_{-\infty}^{a} f(x)\,dx$$



$$\int_{-\infty}^{\infty} f(x)\,dx$$

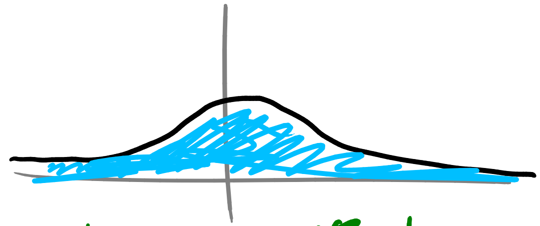
This is defined by splitting it up:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

$$= \left(\lim_{t\to-\infty} \int_t^0 f(x)\,dx\right) + \left(\lim_{t\to\infty} \int_0^t f(x)\,dx\right)$$

[If <u>either</u> of these lim. does not exist, we say the integral is <u>divergent</u>; otherwise it's <u>convergent</u>]

Ex $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$



$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \qquad \left(\text{also } = 2\int_0^\infty \frac{1}{1+x^2}\,dx\right)$$

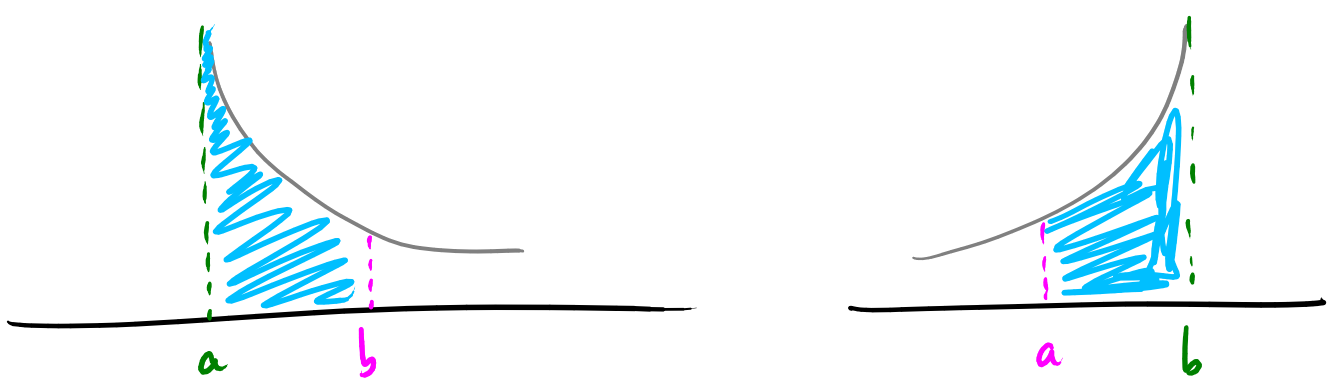
$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \tan^{-1}x \Big|_t^0 + \lim_{t\to\infty} \tan^{-1}x \Big|_0^t$$

$$= \lim_{t\to-\infty} -\tan^{-1}t + \lim_{t\to\infty} \tan^{-1}t$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \underline{\underline{\pi}}$$

<u>Another kind of imp. $\int$ we saw last time:</u>



Ex $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$: <u>improper</u> b/c $\frac{1}{\sqrt{x-2}}$ goes to ∞ as $x\to 2^+$.

So to define this int:

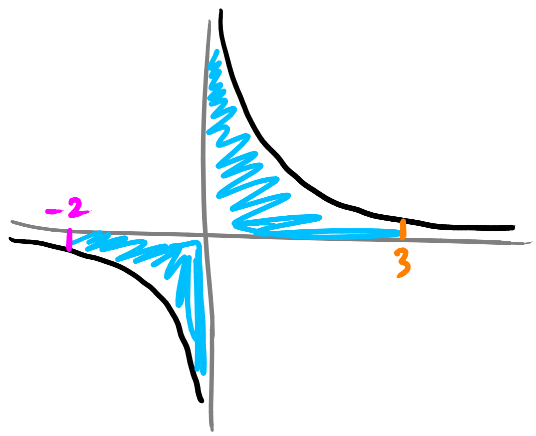
$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t\to 2^+} \left(2\left(\sqrt{3}-\sqrt{t-2}\right)\right)$$

(get this by u-sub: $u = x-2$)

$$= 2\sqrt{3} \quad (\underline{\text{convergent}})$$

Ex $\displaystyle\int_{-2}^{3} \frac{1}{x}\,dx$



Improper b/c of vert. asymp. at $x=0$

$$\int_{-2}^{3} \frac{1}{x}\,dx = \int_{-2}^{0} \frac{1}{x}\,dx + \int_{0}^{3} \frac{1}{x}\,dx$$

$$= \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x}\,dx + \lim_{t\to 0^+} \int_{t}^{3} \frac{1}{x}\,dx$$

$$= \lim_{t\to 0^-} \left(\ln|x|\Big|_{-2}^{t}\right) + \lim_{t\to 0^+} \left(\ln x\Big|_{t}^{3}\right)$$

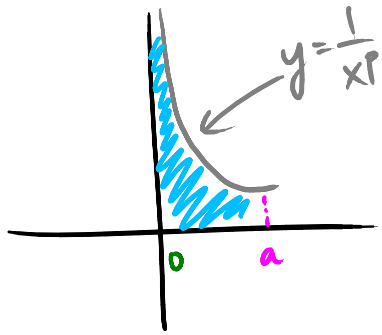
$$= \underbrace{\lim_{t\to 0^-} \left(\ln|t| - \ln 2\right)}_{\downarrow\ -\infty} + \underbrace{\lim_{t\to 0^+} \left(\ln 3 - \ln|t|\right)}_{\downarrow\ -\infty}$$

so neither of these limits goes to a finite #:

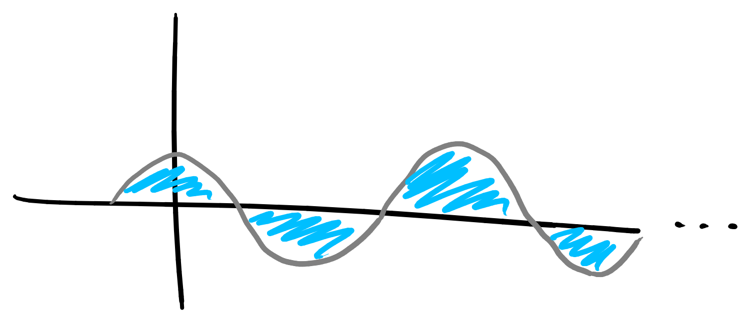
$\int$ is divergent!

A general rule:

$$\int_0^a \frac{1}{x^p}\,dx \quad \text{is:} \quad \begin{array}{l} \underline{\text{convergent}} \text{ if } p<1 \\ \underline{\text{divergent}} \text{ if } p\geq 1 \end{array}$$



Ex $\displaystyle\int_0^\infty \cos x\,dx$



$$= \lim_{t\to\infty} \int_0^t \cos x\,dx$$

$$= \lim_{t\to\infty} \sin x \Big|_0^t = \lim_{t\to\infty} \sin t \quad \underline{\text{does not exist}}$$

i.e. $\displaystyle\int_0^\infty \cos x\,dx$ diverges

Lecture 24 | 22 Mar 2010

Housekeeping: Exam 2 Tue Apr 6 7-9pm WEL 1.316

Last time: improper integrals

So far we always looked at functions $f(x)$ ← only one variable.

In reality things often depend on more than one variable.

e.g. the heat index depends on temperature and humidity.
$f(x,y)$ x y

or the elevation depends on longitude and latitude

the price of cheese depends on supply and demand

Partial derivatives (Ch 15.3)

If we have $f(x,y)$ we can ask:

- How does f change if we vary x a little while holding y constant?

Define $\dfrac{\partial f}{\partial x} = \lim_{h \to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$

[To calculate it, just treat y as const and differentiate like usual with resp to x.]

Ex If $f(x,y) = x^2 \sin(y)$ then $\dfrac{\partial f}{\partial x} = 2x \sin(y)$

Ex If $f(x,y) = x^2 \sin(y)$ then $\dfrac{\partial f}{\partial x}$ at $(x,y) = (1, \pi/2)$ is $2 \cdot 1 \cdot 1 = 2$

Similarly $\dfrac{\partial f}{\partial y} = \lim\limits_{h \to 0} \dfrac{f(x, y+h) - f(x,y)}{h}$

(hold x constant and differentiate w/respect to y).

__Ex__ Say $f(x,y) = 4x^2y + 7\sin(x)$

$$\frac{\partial f}{\partial x} = \underline{\underline{8xy + 7\cos(x)}} \qquad \frac{\partial f}{\partial y} = \underline{\underline{4x^2}}$$

Can also look at __2nd deriv__:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \underline{\underline{8y - 7\sin(x)}} \qquad \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \underline{\underline{8x}}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(4x^2) = \underline{\underline{8x}} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(4x^2) = \underline{\underline{0}}$$

__Notice__: $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$! This always happens (whenever both are continuous)

Ex $f(x,y) = \sin(xy)$

$$\frac{\partial f}{\partial x} = y\cos(xy) \qquad\qquad \frac{\partial f}{\partial y} = x\cos(xy)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(y\cos(xy)\right) = y \cdot y(-\sin(xy)) = -y^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(x\cos(xy)\right) = x \cdot x(-\sin(xy)) = -x^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\cos(xy)\right) = 1\cdot\cos(xy) + x\cdot\left(-y\sin(xy)\right)$$

$$= \underline{\underline{\cos(xy) - xy\sin(xy)}}$$

[Also, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \cos(xy) - xy\sin(xy)$]

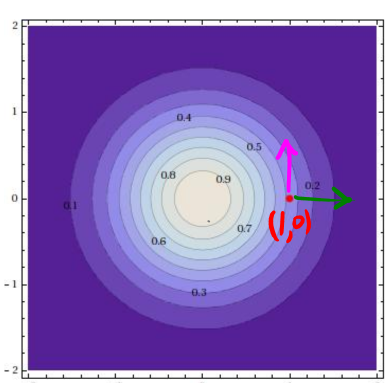
because $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$

Picturing the partial derivs:

Say $f = e^{-x^2-y^2}$ $\qquad \frac{\partial f}{\partial x} = -2x\,e^{-x^2-y^2} \qquad \frac{\partial f}{\partial y} = -2y\,e^{-x^2-y^2}$

So at $(x,y) = (1,0)$: $\frac{\partial f}{\partial x} = -\frac{2}{e} < 0 \qquad \frac{\partial f}{\partial y} = 0$

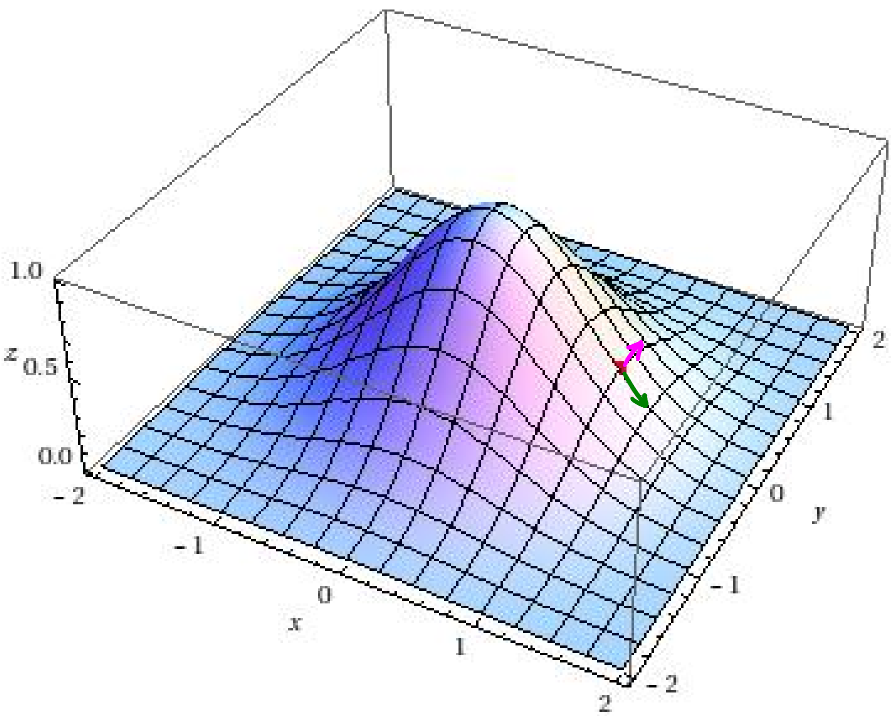
See this from the "contour map" of f:
(lighter color = larger value of f)



$\longrightarrow$: going downhill
i.e. f decreasing
i.e. $\frac{\partial f}{\partial x} < 0$

$\uparrow$: neither uphill nor downhill
i.e. $\frac{\partial f}{\partial y} = 0$

3-d picture of f: (height = f(x,y))



$\rightarrow$ downhill

$\uparrow$ neither uphill nor downhill

<u>Notation</u>: Write f_x for $\dfrac{\partial f}{\partial x}$

f_y for $\dfrac{\partial f}{\partial y}$

f_{xx} for $\dfrac{\partial^2 f}{\partial x^2}$

f_{xy} for $\dfrac{\partial^2 f}{\partial x \partial y}$ [so $f_{xy} = f_{yx}$]

f_{yx} for $\dfrac{\partial^2 f}{\partial y \partial x}$

f_{yy} for $\dfrac{\partial^2 f}{\partial y^2}$

[etc. for higher derivatives]

Lecture 25 | 24 Mar 2010

Last time: <u>partial derivatives</u>

$$\begin{tikzcd}
 & & \frac{\partial^2 f}{\partial x^2} \\
f(x,y) \arrow[r, squiggly] \arrow[dr, squiggly] & \frac{\partial f}{\partial x} \arrow[ur, squiggly] \arrow[r, squiggly] & \frac{\partial^2 f}{\partial x \partial y} \\
 & \frac{\partial f}{\partial y} \arrow[r, squiggly] \arrow[dr, squiggly] & \frac{\partial^2 f}{\partial y \partial x} \\
 & & \frac{\partial^2 f}{\partial y^2}
\end{tikzcd}$$

these two are <u>equal</u>

<u>Notation</u>:

$$\frac{\partial f}{\partial x} = f_x \qquad \frac{\partial f}{\partial y} = f_y$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \quad \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$f_{xy} = f_{yx}$$

<u>Terminology</u>: If $f(x,y) =$ productivity

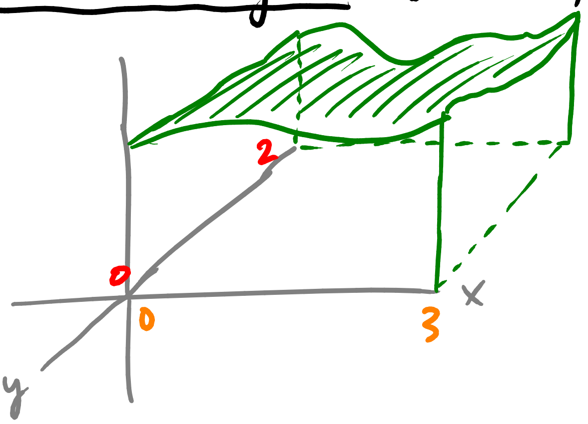
where $x =$ capital

$y =$ labor

Then f_x is called <u>marginal productivity</u> of capital

f_y is " " of labor

Iterated integrals (Ch 16.2)



Surface $z=f(x,y)$ [like $y=f(x)$]

In 1-variable calc, we studied area under the curve $y=f(x)$

Now study volume under the surface $z=f(x,y)$

cross section area

Cut by planes at fixed y: $V=\int_0^2 A(y)\,dy$

$A(y)=\int_0^3 f(x,y)\,dx$ gives the cross section [as usual]

So $V=\int_0^2\left[\int_0^3 f(x,y)\,dx\right]dy$

Ex Suppose $f(x,y)=4xy+3x^2$

Then $V=\int_0^2\left[\int_0^3 4xy+3x^2\,dx\right]dy$

$$=\int_0^2\left[2x^2y+x^3\Big|_{x=0}^{x=3}\right]dy$$

$$=\int_0^2(18y+27-0)\,dy$$

$$=9y^2+27y\Big|_{y=0}^{y=2}$$

$$=36+54=\underline{\underline{90}}$$

We could also try doing the $\int$'s in the other order:

$$V = \int_0^3 \left[\int_0^2 4xy + 3x^2 \, dy \right] dx$$

$$= \int_0^3 \left[2xy^2 + 3x^2 y \Big|_{y=0}^{y=2} \right] dx$$

$$= \int_0^3 8x + 6x^2 \, dx$$

$$= 4x^2 + 2x^3 \Big|_{x=0}^{x=3}$$

$$= 36 + 54 = \underline{\underline{90}}$$

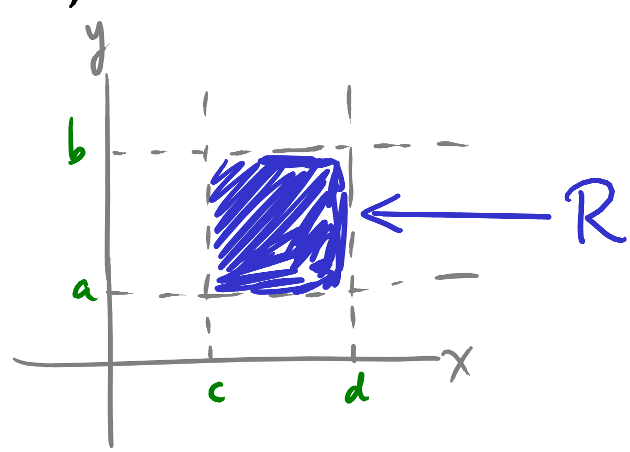
Notice, answer doesn't depend on the order:

$$\int_a^b \left[\int_c^d f(x,y)\, dx \right] dy = \int_c^d \left[\int_a^b f(x,y)\, dy \right] dx$$

("Fubini's Theorem")

We also write it as

$$\iint_R f(x,y)\, dA$$



Ex If $R = \{1 \le x \le 2,\ 0 \le y \le \pi\}$ and $f(x,y) = y \sin(xy)$

What is $\iint_R f(x,y)\, dA$?

It is $\int_0^\pi \left[\int_1^2 y \sin(xy)\, dx \right] dy$

(or: $\int_1^2 \left[\int_0^\pi y \sin(xy)\, dy \right] dx$, but that's harder to calculate)

$$= \int_0^\pi \left(-\cos(xy) \Big|_{x=1}^{x=2} \right) dy$$

$$= \int_0^\pi \left(-\cos(2y) + \cos(y) \right) dy$$

$$= -\frac{1}{2}\sin(2y) + \sin(y) \Big|_{y=0}^{y=\pi}$$

$$= \underline{\underline{0}}$$

Ex Find the volume of the solid which lies under the graph of

$$z = f(x,y) = 4 + x^2 - y^2$$

and over the rectangle $\left.\begin{array}{l} -1 \leq x \leq 1 \\ 0 \leq y \leq 2 \end{array}\right]$ (call this rectangle R)

$$V = \iint_R f(x,y)\, dA$$

$$= \iint_R 4 + x^2 - y^2\, dA$$

$$= \int_{-1}^{1} \left[\int_0^2 4 + x^2 - y^2\, dy \right] dx$$

$$= \int_{-1}^{1} \left[4y + x^2 y - \frac{1}{3} y^3 \Big|_{y=0}^{y=2} \right] dx$$

$$= \int_{-1}^{1} \left[\left(8 + 2x^2 - \frac{8}{3}\right) - 0 \right] dx$$

$$= \int_{-1}^{1} \frac{16}{3} + 2x^2\, dx$$

$$= \frac{16}{3} x + \frac{2}{3} x^3 \Big|_{-1}^{1} = 6 + 6 = \underline{\underline{12}}$$

Lecture 26

25 Mar 2010

Housekeeping: Guest lectures next MW (Maria Gualdani)

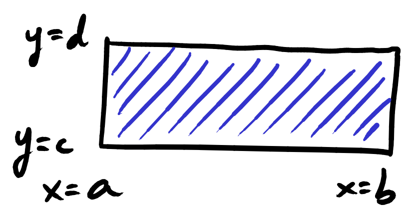
No office hrs next M

Extra office hr next F (11-12 in add^n to usual 10-11)

Exam 2 April 6

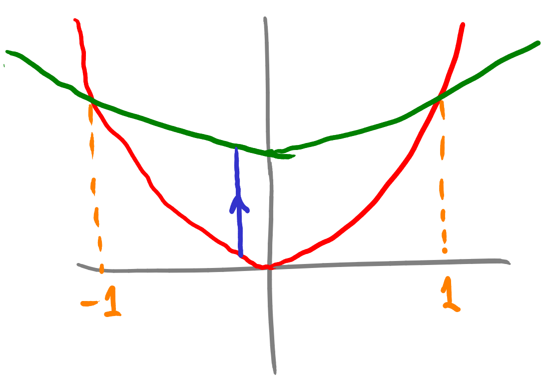
Last time: <u>iterated integrals</u> $\displaystyle\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$

= double integrals over <u>rectangles</u>



How about more complicated domains than rectangles?

<u>Ex</u> Evaluate $\displaystyle\iint_D (x+2y)\,dA$ where D is the domain lying between the curves $y=2x^2$ and $y=1+x^2$.



It's given by an iterated integral

$$\int_{-1}^{1}\left[\int_{2x^2}^{1+x^2} (x+2y)\,dy\right] dx$$

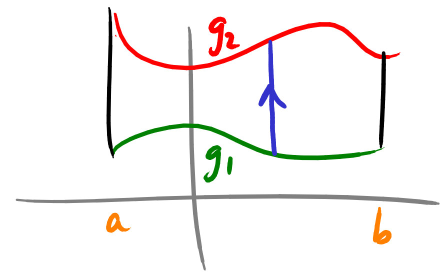
$$= \int_{-1}^{1}\left[\,xy+y^2\,\Big|_{y=2x^2}^{y=1+x^2}\right] dx$$

$$= \int_{-1}^{1}\left[x(1+x^2)+(1+x^2)^2-\left(x(2x^2)+(2x^2)^2\right)\right] dx$$

$$= \cdots = \frac{32}{15}$$

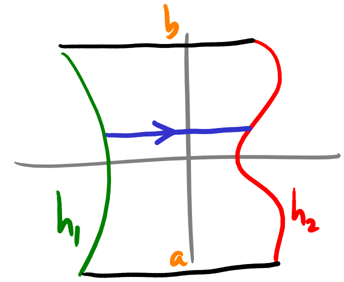
We use this basic method <u>whenever</u> we have a domain of the form

$$D = \{(x,y): \quad a \le x \le b, \quad g_1(x) \le y \le g_2(x)\}$$



$$\iint_D f(x,y)\, dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right] dx$$

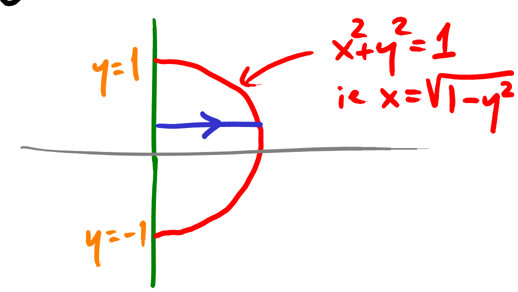
If $D = \{(x,y): \quad a \le y \le b, \quad h_1(y) \le x \le h_2(y)\}$



$$\iint_D f(x,y)\, dA = \int_a^b \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right] dy$$

<u>Ex</u> $\displaystyle\iint_D xy^2\, dA$

D is the domain enclosed by
the line $x=0$
the circle $x^2+y^2=1$
with $x>0$



$$= \int_{-1}^{1} \left[\int_0^{\sqrt{1-y^2}} xy^2\, dx \right] dy$$

$$= \int_{-1}^{1} \left[y^2 \frac{x^2}{2} \Big|_{x=0}^{x=\sqrt{1-y^2}} \right] dy$$

$$= \int_{-1}^{1} \left[\tfrac{1}{2} y^2 (1-y^2) - 0 \right] dy \quad = \cdots = \frac{2}{15}$$

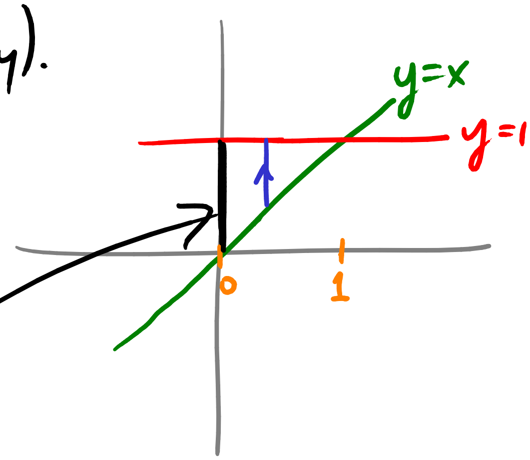
Ex $\displaystyle\int_0^1 \left[\int_x^1 \sin(y^2)\, dy \right] dx$

This looks hard (can't do the inside $\int$ over y).

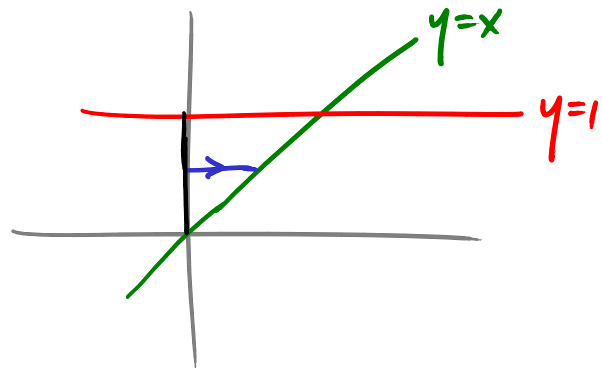
Interpret it as a double $\int$:

$$\iint_D \sin(y^2)\, dA$$

where D is this triangle



Do it by integrating over x first:



$$\int_0^1 \left[\int_0^y \sin(y^2)\, dx \right] dy$$

$$= \int_0^1 \left[x \sin(y^2) \Big|_{x=0}^{x=y} \right] dy$$

$$= \int_0^1 y \sin(y^2)\, dy$$

$$= -\frac{1}{2} \cos(y^2) \Big|_{y=0}^{y=1}$$

$$= \underline{\underline{\frac{1}{2}\left(1 - \cos(1)\right)}}$$

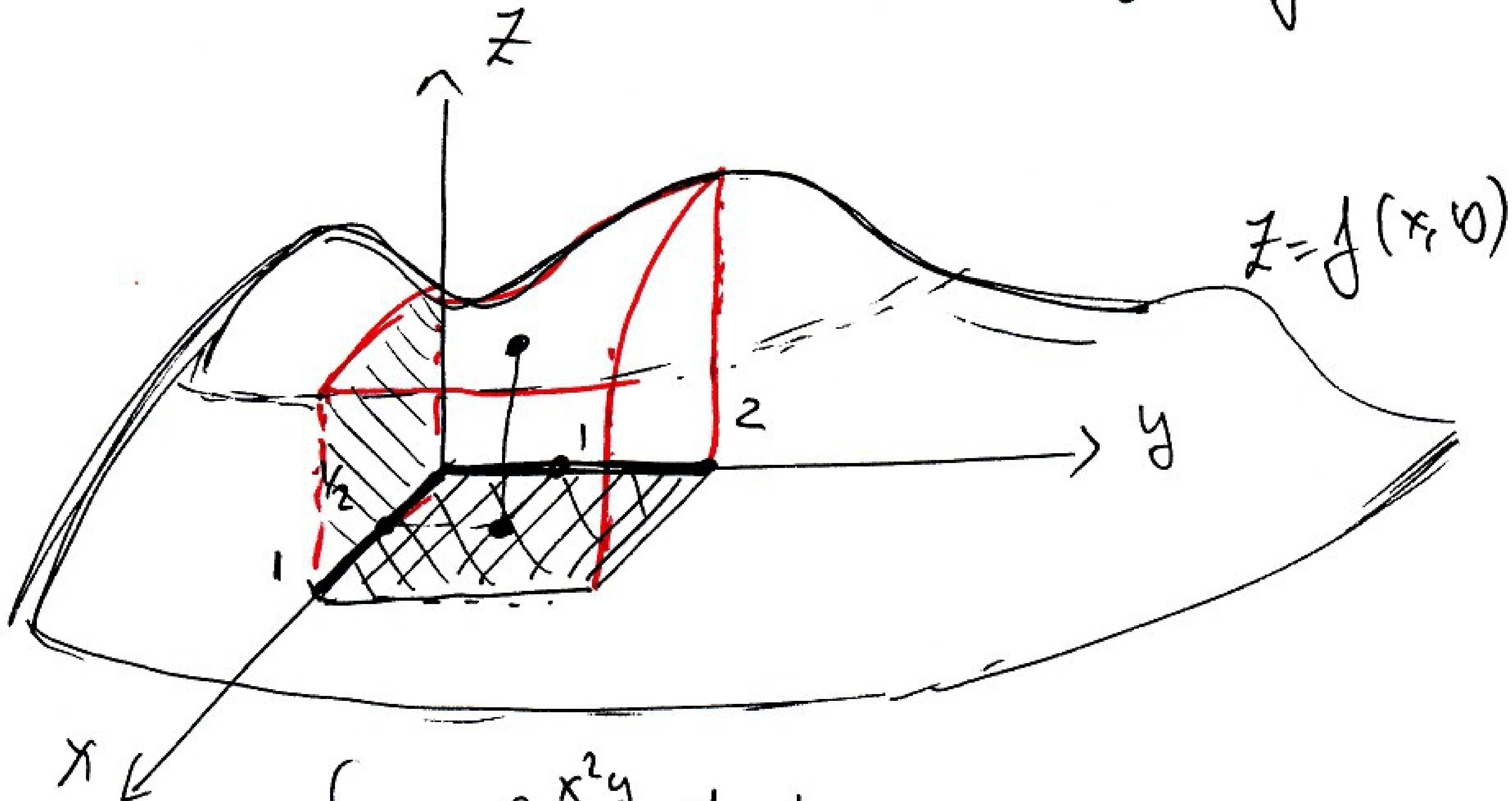
Chapter 16.2 / 16.3

Double Integrals

ex: $z = xy\,e^{x^2 y}$

integration domain

$\mathcal{D} = \{ (x, y)$ such that $0 \le x \le 1$, $0 \le y \le 2 \}$



$$\int_{\mathcal{D}} xy\,e^{x^2 y}\,dx\,dy =$$

$$\underbrace{\int_0^2 \left(\int_0^1 xy\,e^{x^2 y}\,dx \right)}_{\text{area}} dy = \int_0^2 y \left(\int_0^1 x e^{x^2 y}\,dx \right) dy$$

Compute

$$\int_0^1 x e^{x^2 y}\, dx = \int \frac{1}{2y} u_x^{=\frac{d}{dx}u} = (*)$$

$$\frac{d}{dx}\left(e^{x^2 y}\right) = 2xy \cdot e^{x^2 y}$$

$$f' = f_x$$

$$u = e^{x^2}$$

$$\left(e^{x^2 y}\right)_x = \frac{d}{dx}\left(e^{x^2 y}\right)$$

$$\frac{du}{dx} = 2x e^{x^2}$$

$$(*) = \boxed{\frac{1}{2y} \int_0^1 \left(e^{x^2 y}\right)_x dx}$$

$$= \frac{1}{2y}\left[e^{x^2 y} \Big|_0^1 \right] =$$

$$= \frac{1}{2y}\left(e^y - 1\right)$$

$$\Rightarrow \int_0^1 x e^{x^2 y}\, dx = \frac{1}{2y}\left(e^y - 1\right)$$

back to the double integral :

$$\int_0^2 y \left(\underbrace{\int_0^1 x e^{x^2 y}\, dx}_{} \right) dy =$$

$$= \int_0^2 y \left(\frac{1}{2y} \left(e^{y} - 1 \right) \right) dy =$$

$$= \int_0^2 \cancel{y} \cdot \frac{1}{2\cancel{y}} \left(e^{y} - 1 \right) dy =$$

$$= \int_0^2 \frac{1}{2} \left(e^{y} - 1 \right) dy = \frac{1}{2} \int_0^2 \left(e^{y} - 1 \right) dy =$$

$$= \frac{1}{2} \left(e^{y} - y \right) \Big|_0^2 = \frac{1}{2} \left(e^{2} - 2 \right) - \frac{1}{2} \left(1 - 0 \right) =$$

$$= \frac{1}{2} e^{2} - 1 - \frac{1}{2} = \boxed{\frac{1}{2} e^{2} - \frac{3}{2}}$$

$$\int_0^1 x e^{x^2 y} \, dx$$

$$\boxed{= \frac{1}{2y} \int_{*}^{*} e^u \, du}$$

$$u = x^2 y$$

$$du = 2xy$$

$$= \frac{1}{2y} \int_{*}^{*} (e^u)_u \, du$$

$$= \int_1^e x^3 \left(y \Big|_0^{\ln(x)} \right) dx$$

$$= \int_1^e x^3 \left(\ln(x) - 0 \right) dx$$

$$= \int_1^e x^3 \ln(x) \, dx \longrightarrow \text{by parts}$$

$$f(x) = \ln(x) \longrightarrow f'(x) = \frac{1}{x}$$

$$g'(x) = x^3 \longrightarrow g(x) = \frac{x^4}{4}$$

$$\int_a^b f'g = fg\Big|_a^b - \int_a^b fg'$$

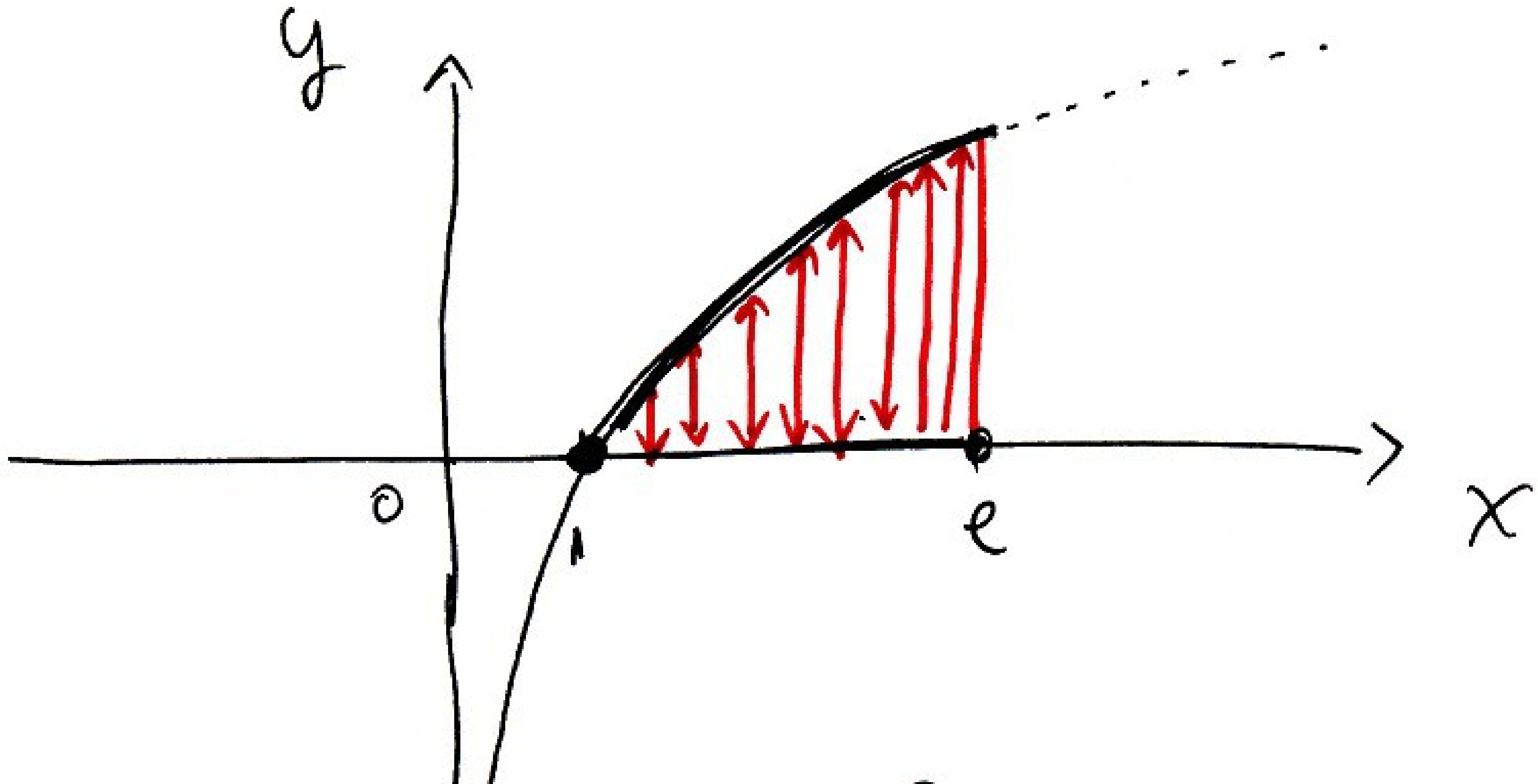
$$\int_1^e x^3 \ln(x) \, dx = \frac{x^4}{4} \cdot \ln(x) \Big|_1^e - \int \frac{1}{4} x^4 \cdot \frac{1}{x} \, dx$$

$$= \frac{x^4 \cdot \ln(x)}{4} \Big|_1^e - \frac{1}{4} \int_1^e x^3 \, dx$$

$$= \frac{e^4 \ln(e)}{4} - \frac{\ln(1)}{4} - \frac{1}{4} \cdot \frac{1}{4} x^4 \Big|_1^e = \ldots$$

ex : $z = x^3$

$\mathcal{D} = \{(x,y) \,/\, 1 \le x \le e \,,\, 0 \le y \le \ln(x)\}$



$$\int_{\mathcal{D}} x^3 \, dx\, dy =$$

$\mathcal{D} \Rightarrow$

$$= \cancel{\int_0^{\ln(x)} \left(\int_1^e dx \right) dy}$$

$$= \int_1^e \left(\int_0^{\ln(x)} x^3 \, dy \right) dx =$$

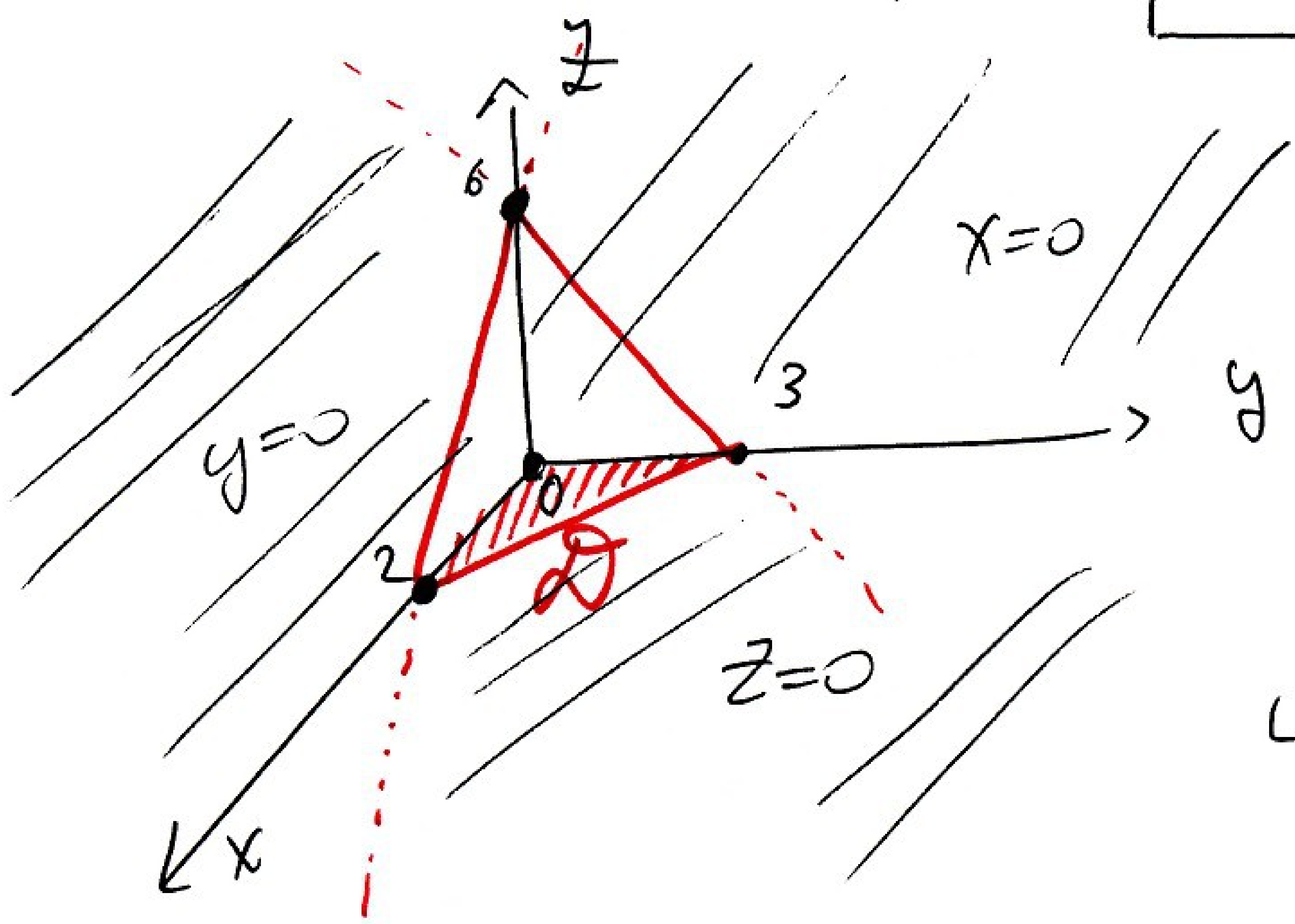
$$= \int_1^e x^3 \left(\int_0^{\ln(x)} dy \right) dx =$$

ex: Volume of a solid $\quad z=0,\ x=0,\ y=0$

bounded by the coordinates planes

and the plane $\boxed{3x + 2y + z = 6}$

$z = 6 - 3x - 2y$



$$\underbrace{\int_{D} 6 - 3x - 2y \; dx\,dy}_{\text{Volume}}$$

$D = \{ (x, y)$ such that

$\leq x \leq$

$\leq y \leq$

March 31st 2010

SEQUENCES

Def: Sequence: list of numbers in order:

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \cdots\cdots \quad a_n \cdots\cdots$

$n: \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots \qquad 100, 101, \cdots,$

Given a general sequence $\{a_n\}_{n \in \mathbb{N}}$,

$\boxed{\text{" } \in \text{ " belongs}}$

ex: take $a_n = 2n$ sequence of even numbers

$0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad \cdots$

$a_{25} = 50$

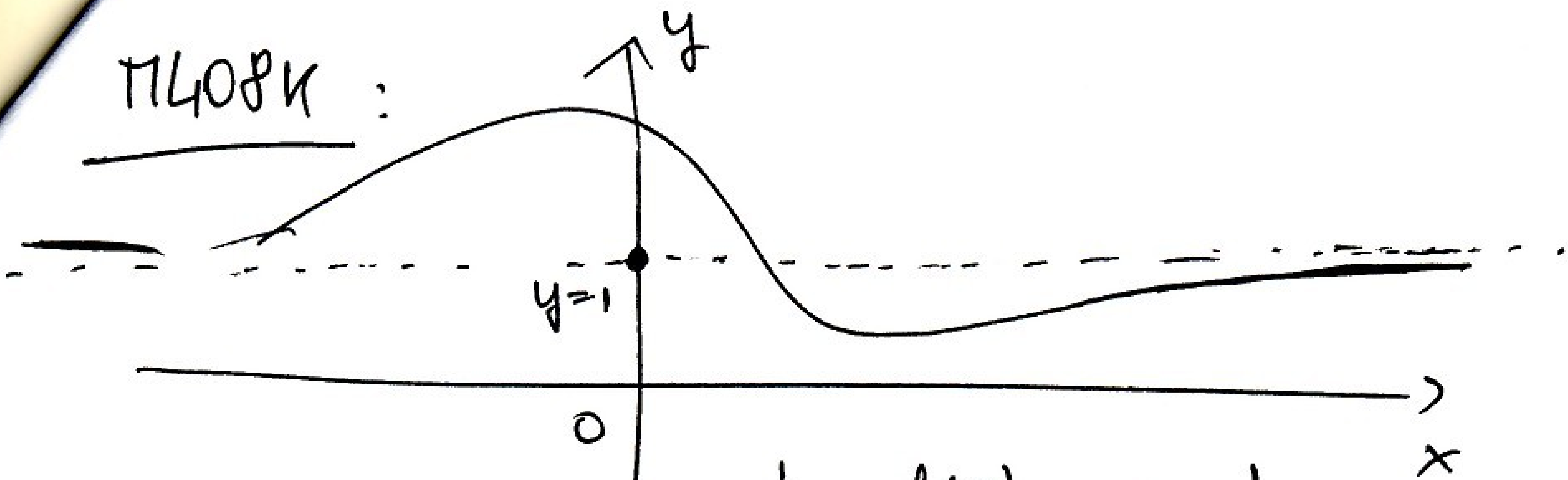
$a_{25} = 2(25) = 50$

$a_n = n^2$

$0 \quad 1 \quad 4 \quad 9 \quad 16 \quad 25 \quad 36$

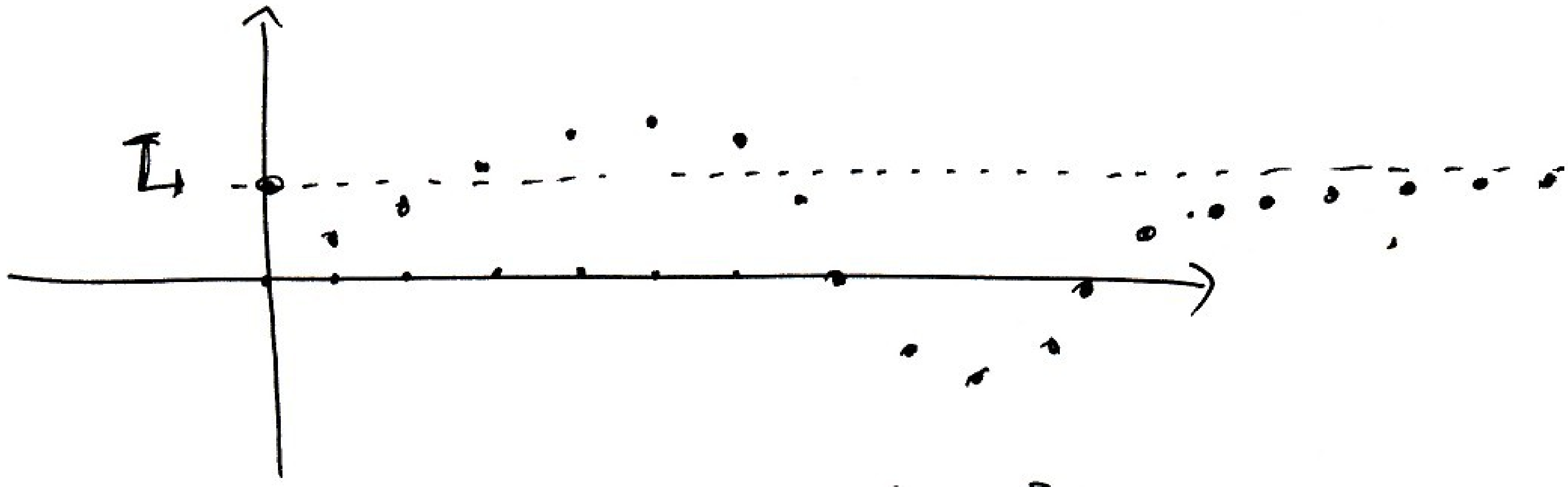
$a_{10} = (10)^2 = 100$

ΠΛ08Κ :

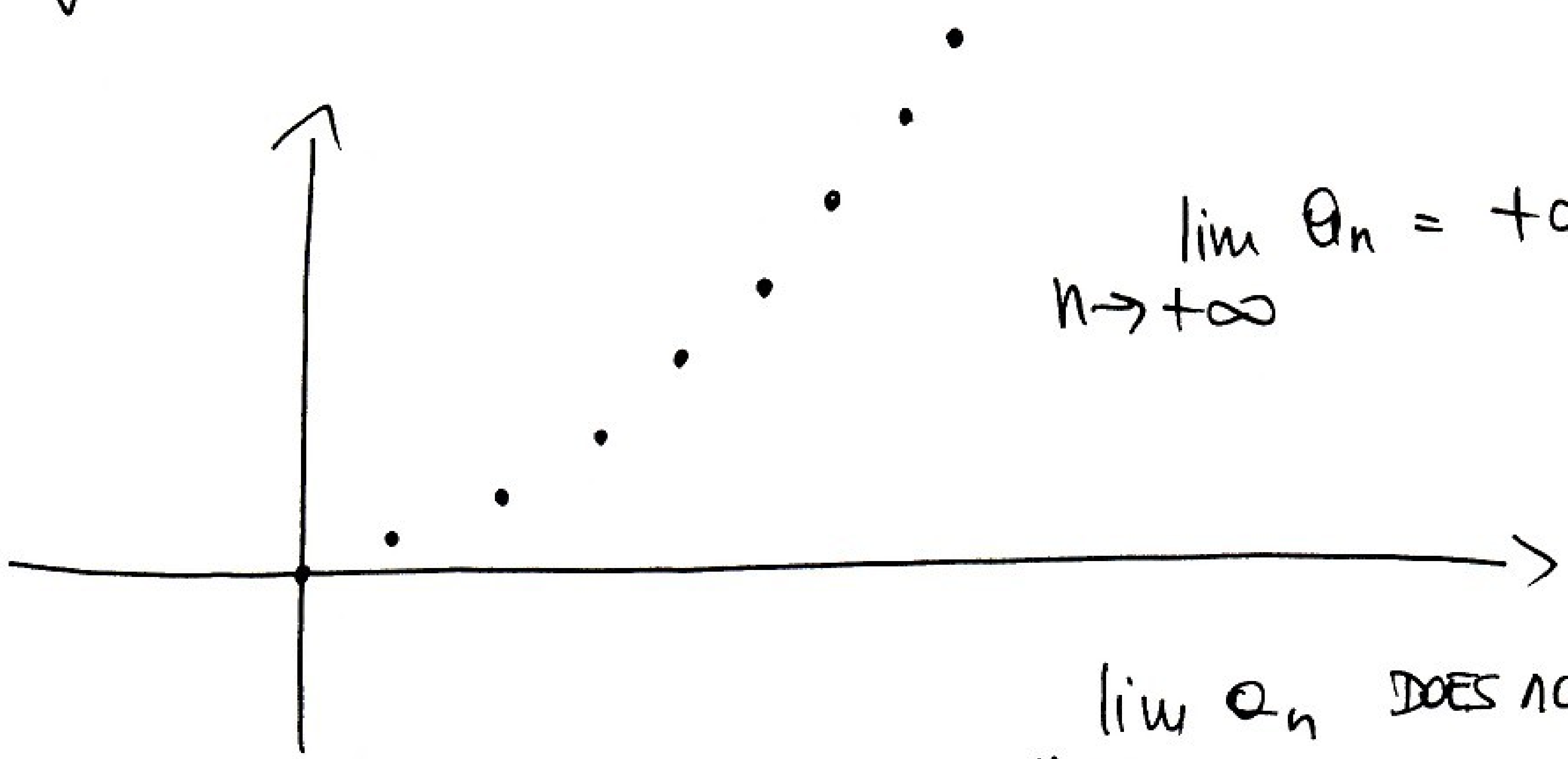


$$\lim_{x \to \pm\infty} f(x) = 1$$

$y = 1$ horizontal asymptote

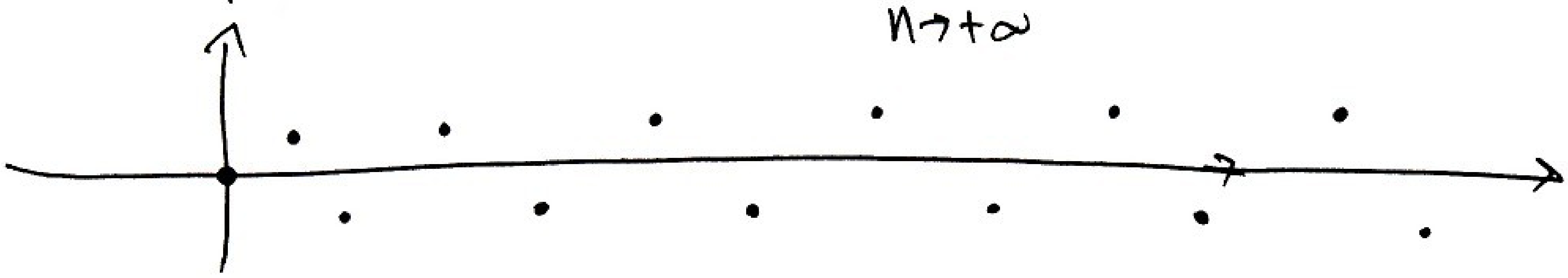


$y = L$ = limit of $\{a_n\}$ as $n \longrightarrow +\infty$



$$\lim_{n \to +\infty} a_n = +\infty$$

$\lim_{n \to +\infty} a_n$ DOES NOT EXIST



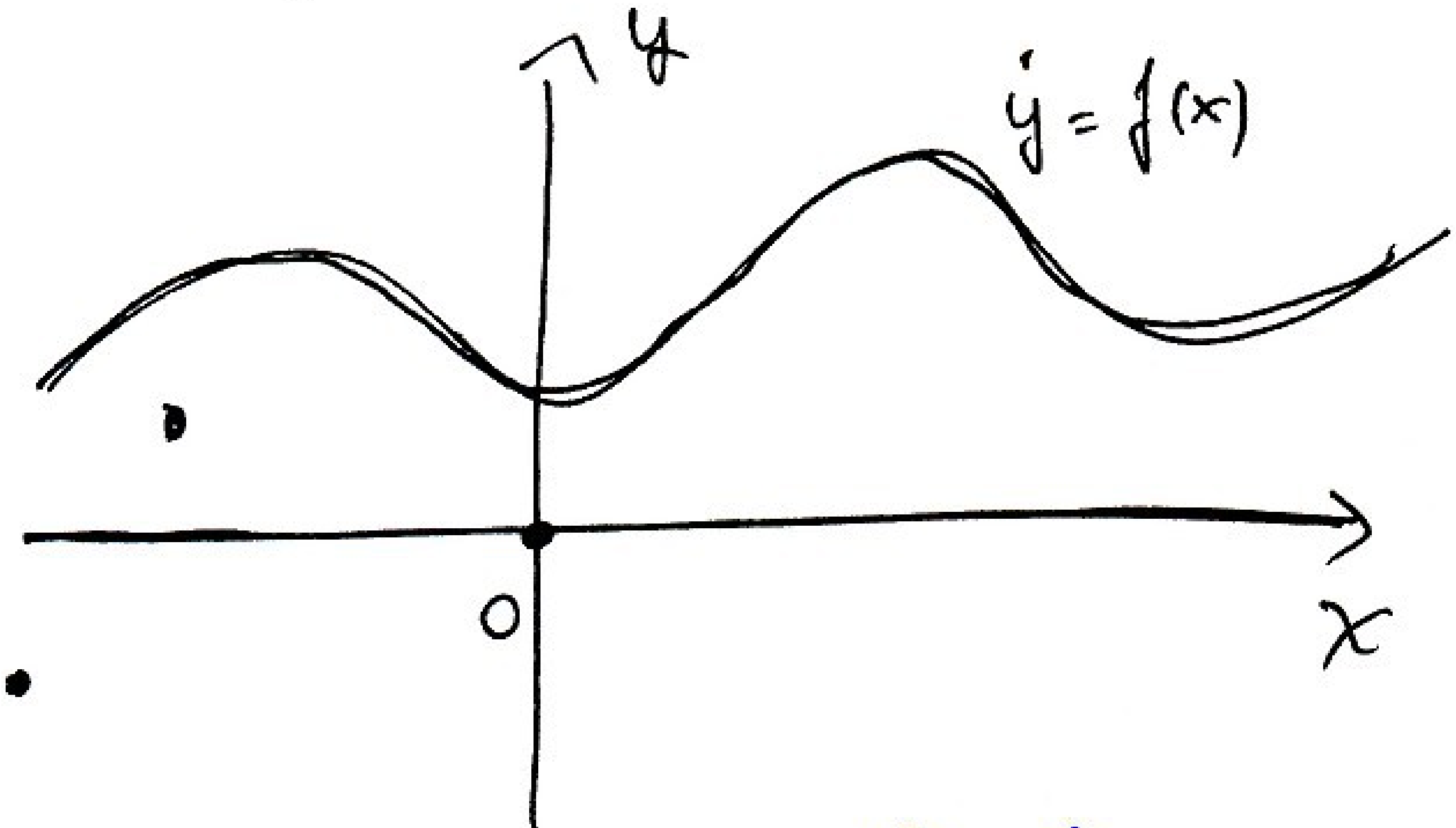
$$a_n = \frac{n}{n+1}$$

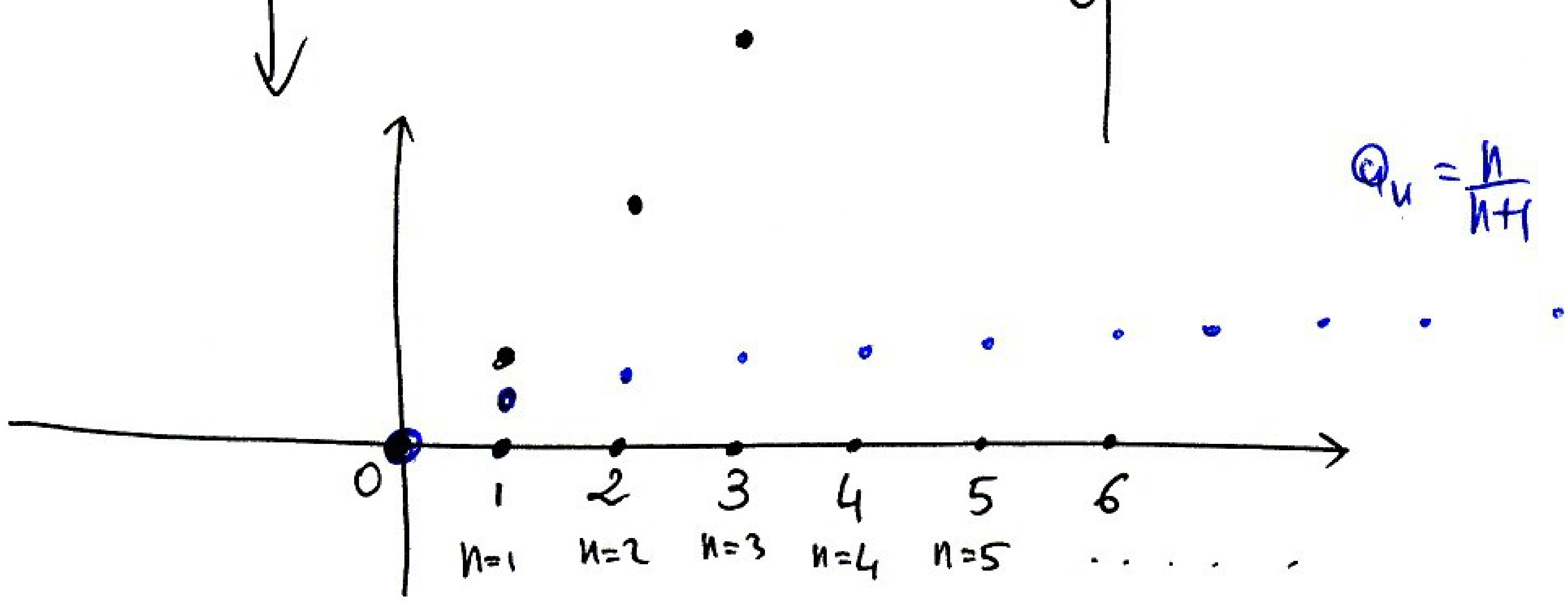
$0 \quad \frac{1}{2} \quad 2/3 \quad 3/4 \quad 4/5 \quad 5/6 \quad 6/7 \quad \ldots\ldots$

$$a_{17} = 17/18$$

Sequences $\Longleftrightarrow$ functions

Can we graph a sequence?





<u>Def</u>: A sequence $\{a_n\}_{n\in\mathbb{N}}$

converges to a limit L if

$$\lim_{n\to+\infty} a_n = L$$

$\{a_n\}$ is CONVERGENT to L

"a_n approaches L as $n \nearrow +\infty$
a_n is close to L as much as you want for $n \to +\infty$".

<u>Def</u>: If such limit L does not exist

$\Rightarrow$ $\{a_n\}_{n\in\mathbb{N}}$ is **DIVERGENT** ex: $a_n = \frac{1}{n}$

$1 \quad \frac{1}{2} \quad 1/3 \quad 1/4 \quad 1/5 \quad 1/6 \quad \cdots\cdots$

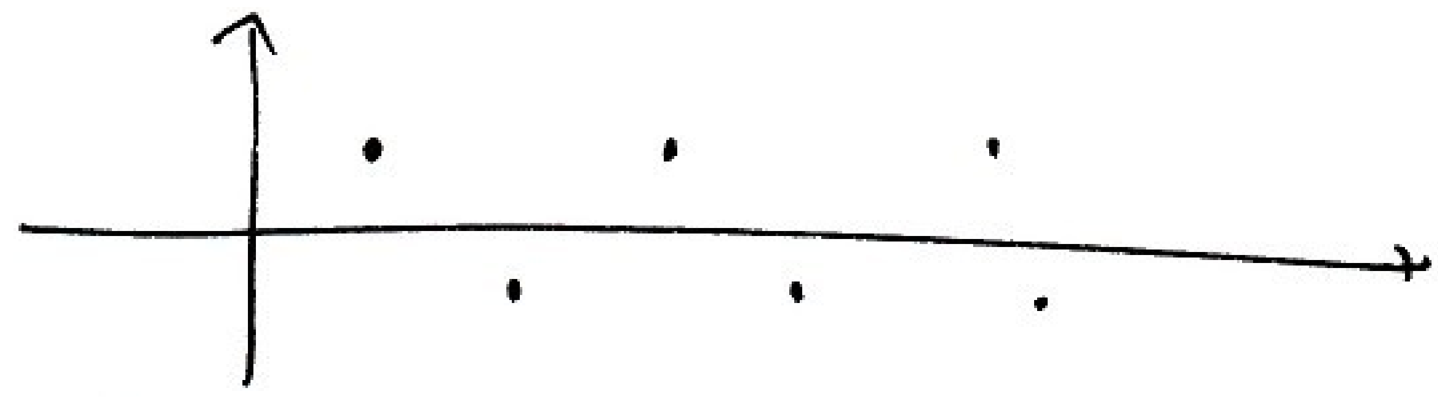
$$\lim_{n\to+\infty} \frac{1}{n} = 0$$

ex: $a_n = \frac{n}{3n+1}$

$$\lim_{n\to+\infty} \frac{\frac{n}{n}}{\frac{3n}{n}+\frac{1}{n}} = \lim_{n\to+\infty} \frac{1}{3+\frac{1}{n}} = 1/3$$

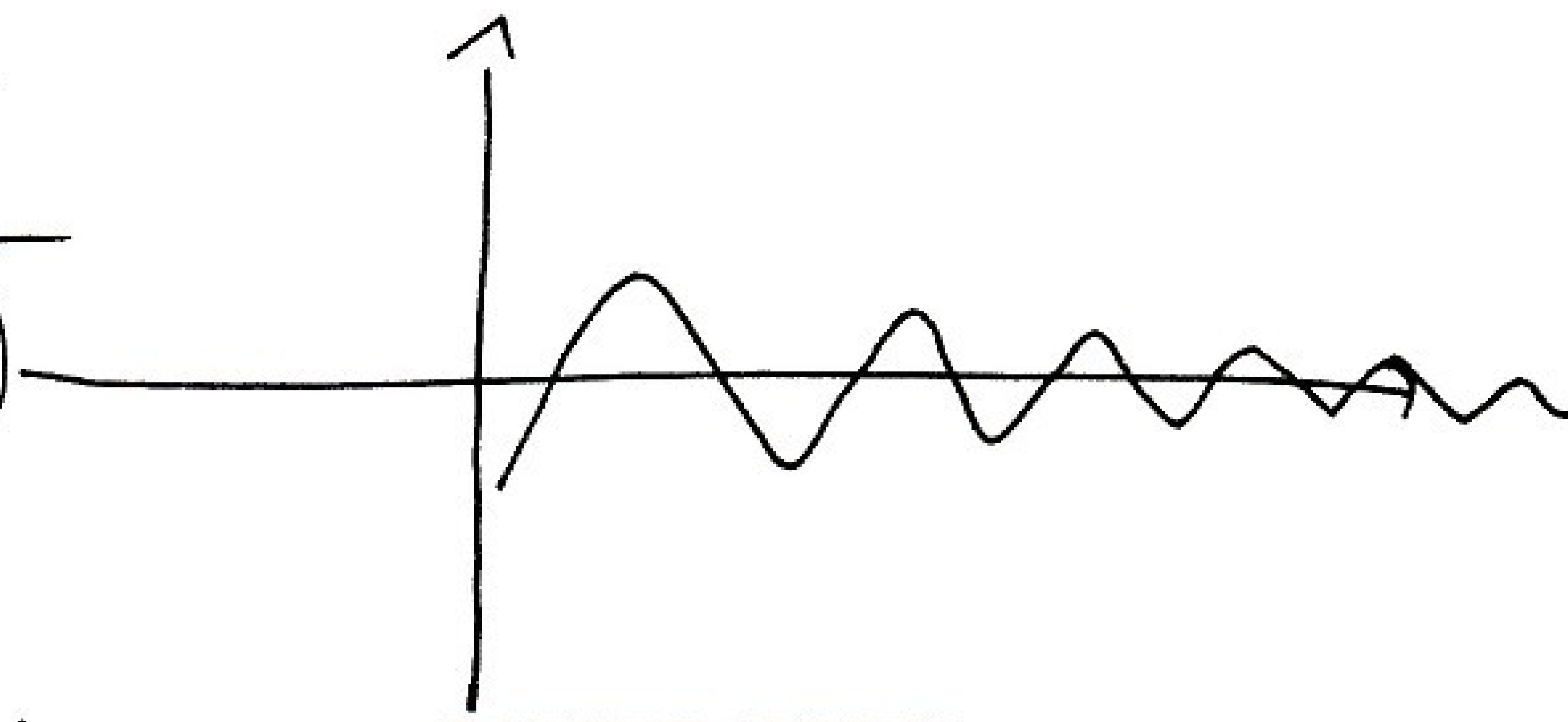
$\Rightarrow \left\{\frac{n}{3n+1}\right\}_{n\in\mathbb{N}}$ is convergent.

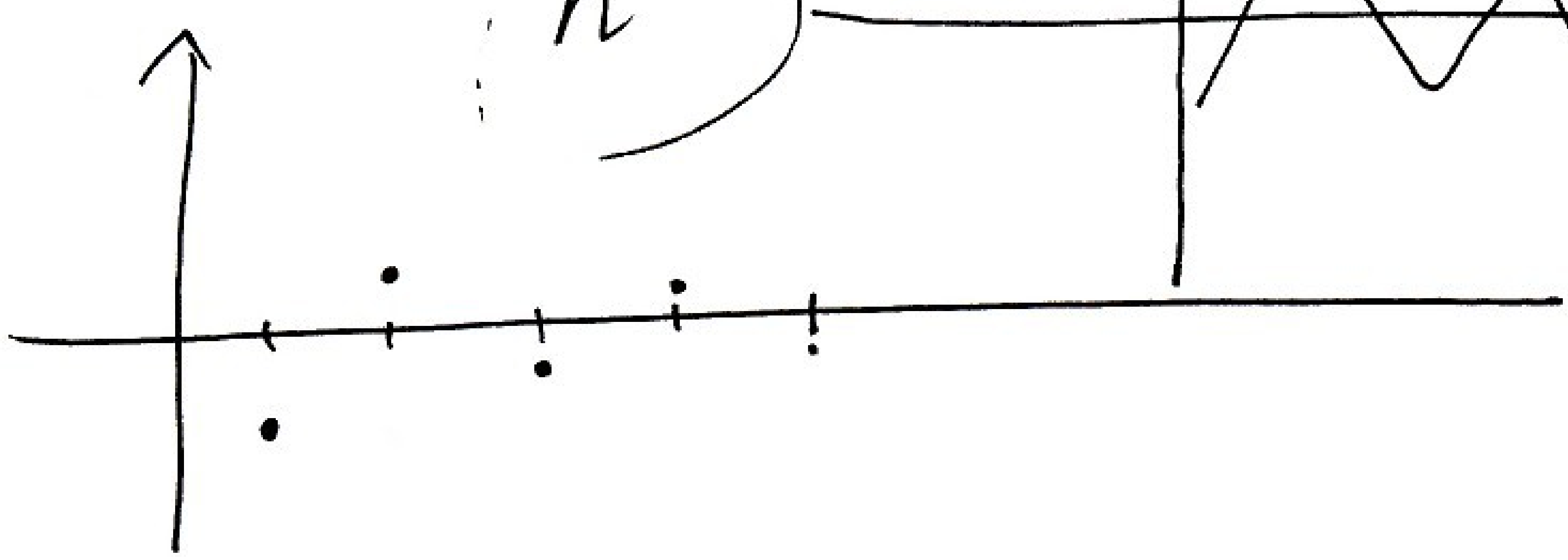
ex: $a_n = (-1)^n$



$\lim_{n\to+\infty} (-1)^n$ does not exist!

ex: $a_n = \frac{(-1)^n}{n^2}$





$$\lim_{n\to+\infty} \frac{(-1)^n}{n^2} = 0$$

ex: $a_n = \frac{\ln(n)}{n}$ ¨̃

Back to functions: $\frac{\ln(x)}{x}$

$$\lim_{x\to+\infty} \frac{\ln(x)}{x} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

$$\Downarrow$$

$$\lim_{n\to+\infty} \frac{\ln(n)}{n} = 0$$

ex:

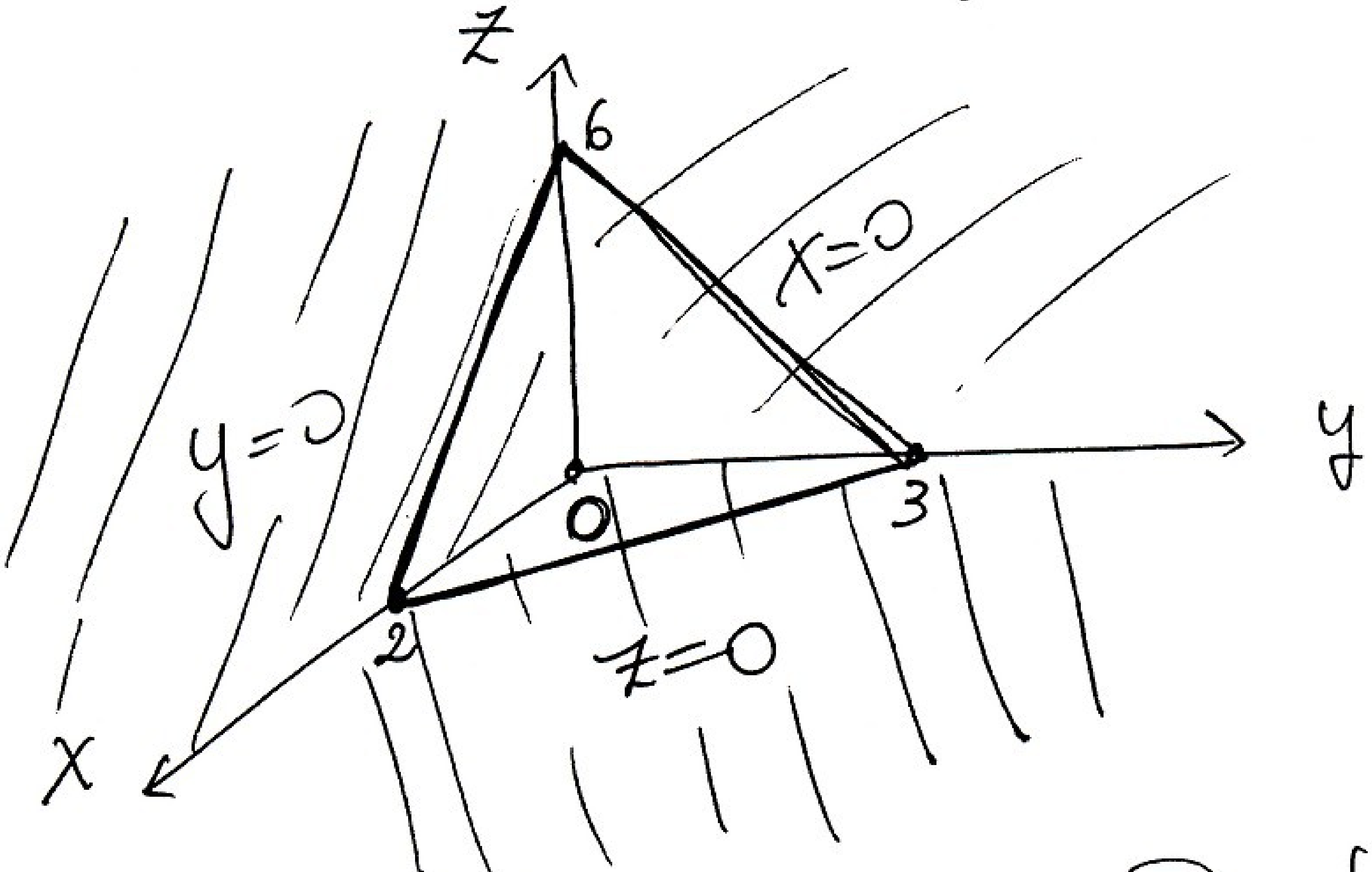
$$a_n = \frac{1}{n^2} \longrightarrow 0$$

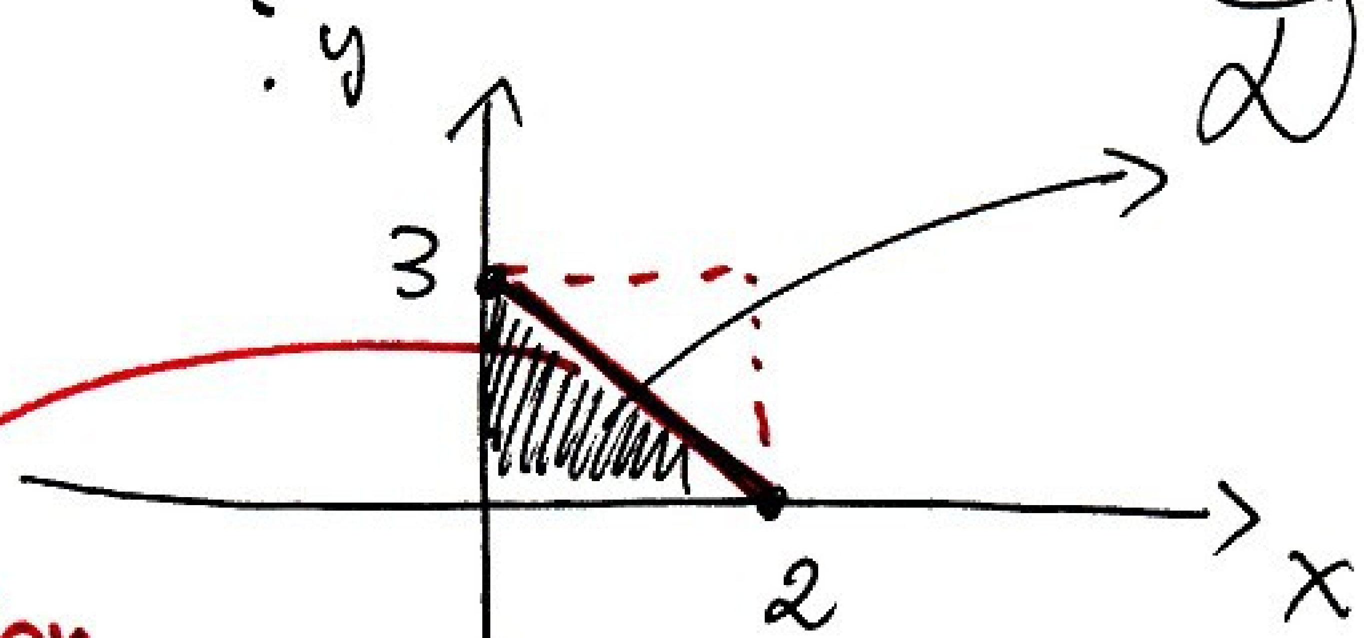
Double Integrals

ex: Volume of the solid bounded by:

(1) coordinate planes $\begin{cases} z=0 \\ y=0 \\ x=0 \end{cases}$

(2) plane $3x+2y+z=6$





$\mathcal{D} = \{(x,y)$ such that
$0 \le x \le 2$
$0 \le y \le -\frac{3}{2}x+$

intersection of plane $3x+2y+z=6$ with "floor" $\equiv (z=0)$

$\Downarrow$

$3x+2y=6 \Rightarrow y = -\frac{3}{2}x+3$

$$\int_{D} \text{"plane"}\, dx\, dy = \int_{D} (6-3x-2y)\, dx\, dy$$

$$3x+2y+z=6$$

$$\Downarrow$$

$$z = 6-3x-2y$$

$$D = \{(x,y) : \quad 0 \le x \le 2$$

$$0 \le y \le -\tfrac{3}{2}x+3$$

$$\int_0^2 \left(\int_0^{-\frac{3}{2}x+3} (6-3x-2y) \quad dy \right) dx$$

$\Rightarrow$ Solve the double integral:

$$\int_0^2 \left(\int_0^{-\frac{3}{2}x+3} (6-3x-2y)\, dy \right) dx =$$

$$= \int_0^2 \left(6y-3xy-y^2 \Big|_0^{-\frac{3}{2}x+3} \right) dx$$

$$= \int_0^2 \left(6\left(-\tfrac{3}{2}x+3\right) - 3x\left(-\tfrac{3}{2}x+3\right) - \left(-\tfrac{3}{2}x+3\right)^2 \right) dx$$

$$= \int_0 \left(-9x + 18 + \frac{9}{2}x^2 - 3x - \frac{9}{4}x^2 - 9 + 9x \right) dx$$

$$= \cdots\cdots \qquad = \underline{\text{NUMBER}}$$

Lecture 29

2 Apr 2010

Housekeeping:

Exam 2 7–9pm April 6 (Tue)

$$\left[\underbrace{7.8}_{\text{indeterm. forms and L'H}},\ \underbrace{8.1\text{–}8.5,\ 8.8}_{\text{integrals}},\ \underbrace{15.3,\ 16.2\text{–}16.3}_{\text{multivar.}},\ \underbrace{12.1}_{\text{seq.}}\right]$$

Review session 7–9pm April 5 (Mon)

Last time: Sequences — ordered list of #'s

$a_1, a_2, a_3, \ldots, a_n, \ldots$

ex. $a_n = n^2(-1)^n$: $-1, 4, -9, 16, -25, \ldots$

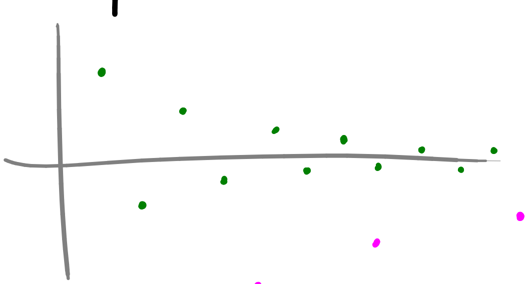
$a_n = \frac{n}{n+1}$: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$a_n = 5$: $5, 5, 5, 5, \ldots$

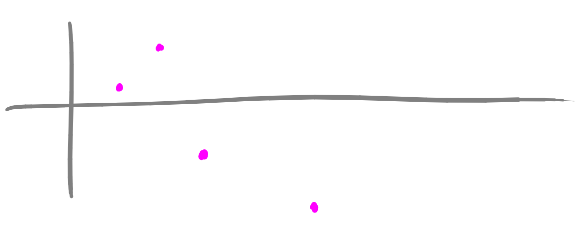
$a_n =$ the n^{th} Fibonacci # : $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$a_n = a_{n-2} + a_{n-1}$

Main question we ask about sequences: do they converge as $n \to \infty$?



$\lim_{n\to\infty} a_n = 0$



a_n diverges

A useful fact: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

<u>Ex</u> Does the sequence $a_n = \left(1-\frac{7}{n}\right)^{-2n}$ <u>converge</u>? If so, to what?

$$\lim_{n\to\infty}\left(1-\frac{7}{n}\right)^{-2n}$$
$$=\left[\lim_{n\to\infty}\left(1-\frac{7}{n}\right)^{n}\right]^{-2}$$
$$=\left[e^{-7}\right]^{-2} = \underline{\underline{e^{14}}} \quad \text{(convergent)}$$

<u>Ex</u> Does $a_n = 2+\frac{1}{n}+\frac{n!+1}{(n+1)!}$ converge? (To what?)

$$= \underset{\downarrow \atop 2}{2} + \underset{\downarrow \atop 0}{\frac{1}{n}} + \boxed{\frac{n!}{(n+1)!}} + \underset{\downarrow \atop 0}{\frac{1}{(n+1)!}}$$

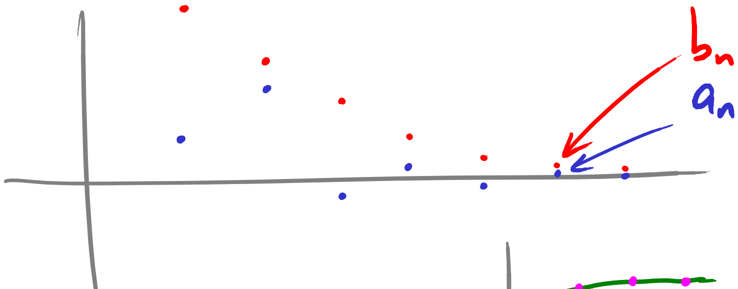
$\left[\text{because } (n+1)! \to \infty\right]$

<u>factorial</u>: $k!$ means $1\cdot 2\cdot 3\cdot(\cdots)\cdot k$ (e.g. $4! = 1\cdot 2\cdot 3\cdot 4 = 24$)

to deal with this term, write it as
$$\frac{1\cdot 2\cdot 3\cdot \cdots \cdot n}{1\cdot 2\cdot 3\cdot \cdots \cdot n\cdot(n+1)} = \frac{1}{n+1} \longrightarrow 0$$

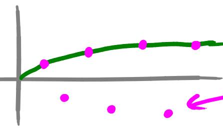
so $\lim_{n\to\infty} a_n = \underline{\underline{2}}$

Another useful fact: if $|a_n| < b_n$ and $b_n \to 0$



then $a_n \to 0$ also!

("Squeeze Theorem")



<u>Ex</u> $a_n = \dfrac{(\tan^{-1} n) \cdot (-1)^n}{\sqrt{n}}$

has $\lim\limits_{n\to\infty} a_n = 0$: <u>intuitively</u> because the <u>top</u> stays bounded as $n \to \infty$, but the <u>bottom</u> goes $\to \infty$ as $n \to \infty$.

Or, to <u>prove</u> it: $\left| (\tan^{-1} n) \cdot (-1)^n \right| = \left| \tan^{-1} n \right| \leq \dfrac{\pi}{2}$

so $|a_n| \leq \dfrac{\pi/2}{\sqrt{n}}$

Call $b_n = \dfrac{\pi}{2} \cdot \dfrac{1}{\sqrt{n}}$. $\lim\limits_{n\to\infty} b_n = 0$

<u>and</u> $|a_n| \leq b_n$

so $\lim\limits_{n\to\infty} a_n = 0$

Ex $\lim_{n\to\infty} \frac{n!}{n^n} = ?$

Rewrite it as $\dfrac{1\cdot 2\cdot 3\cdot 4\cdot \cdots \cdot n}{\underbrace{n\cdot n\cdot n\cdot n\cdot \cdots \cdot n}_{n \text{ times}}}$ e.g. $4^4 = 4\cdot 4\cdot 4\cdot 4$

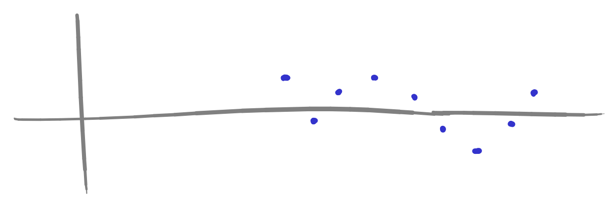
$$= \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdot \cdots \cdot\frac{n}{n}$$

$$= \frac{1}{n}\cdot\left(\text{something} < 1\right)$$

$$< \frac{1}{n} \longrightarrow 0$$

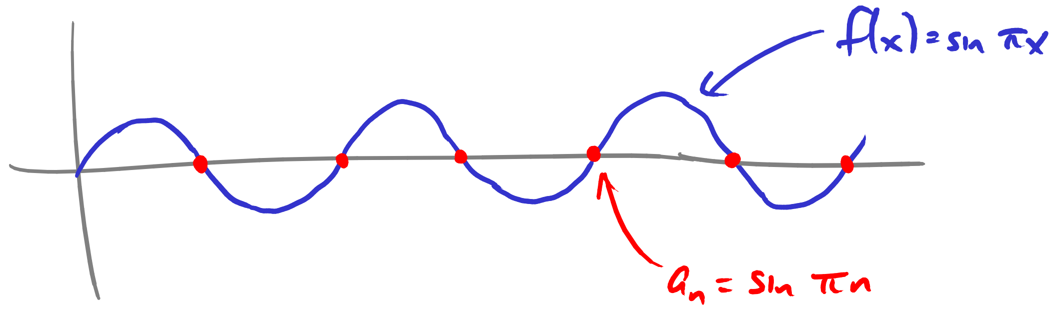
So $\lim_{n\to\infty} \frac{n!}{n^n} = \underline{\underline{0}}$

$a_n = \dfrac{n^2\cos(n)}{1+n^2}$. Intuition: $\sim \cos(n)$ for big n



$\longrightarrow$ <u>divergent</u>

$a_n = \sin(\pi n)$ $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = \underline{\underline{0}}$



NB: $\{a_n\}$ has a limit as $n \to \infty$
even though $f(x) = \sin(\pi x)$ doesn't have a limit as $x \to \infty$!

$$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos(0) = \underline{\underline{1}}$$

$$\lim_{n\to\infty} \ln(2n^2+1) - \ln(n^2+1) = \lim_{n\to\infty} \ln \frac{2n^2+1}{n^2+1} = \underline{\underline{\ln 2}}$$

Fact:

$r=2$: $2, 4, 8, 16, \ldots$ divergent

$r=\frac{1}{2}$: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ convergent ($\to 0$)

$$\lim_{n\to\infty} r^n = \begin{cases} 0 \text{ if } -1<r<1 \\ 1 \text{ if } r=1 \\ \text{does not exist (divergent) if } |r|>1 \end{cases}$$

Lecture 30

Apr 5 2010

Exam 7–9p tomorrow WEL 1.316

Topics:

- $\int$ by parts
- Trigonometric integrals
- Trigonometric substitutions
- Partial fractions
- Indeterminate forms and L'H rule
- Improper integrals
- Partial derivatives
- Iterated integrals
- Double integrals
- Sequences

Trig substitution

general pattern:

$\sqrt{a^2-x^2}$: try $x = a\sin\theta$

$\sqrt{x^2-a^2}$: try $x = a\sec\theta$

$\sqrt{x^2+a^2}$: try $x = a\tan\theta$

Ex $$\int_{3/2}^{3} \frac{\sqrt{9-x^2}}{x^2}\,dx$$

try $x = 3\sin\theta$ $dx = 3\cos\theta\,d\theta$

$x = \frac{3}{2} = 3\sin\theta$

$\to \sin\theta = \frac{1}{2}$

$\to \theta = \frac{\pi}{6}$

$x = 3 \to \sin\theta = 1$

$\to \theta = \frac{\pi}{2}$

then $$= \int_{\pi/6}^{\pi/2} \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\cdot 3\cos\theta\,d\theta$$

$$= \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cos\theta\,d\theta$$

$\sin^2\theta + \cos^2\theta = 1$

$$= \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cos\theta\,d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int_{\pi/6}^{\pi/2} \cot^2\theta\,d\theta$$

$$= \int_{\pi/6}^{\pi/2} (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta \Big|_{\pi/6}^{\pi/2} = \underline{\underline{\sqrt{3} - \frac{\pi}{3}}}$$

Partial fractions

Start with $\int \frac{\text{polynomial}}{\text{polynomial}}\, dx$ and reduce it to, say,

$$\int \frac{3x+4}{x^2+1} + \frac{7}{x-4} + \frac{3}{x+2}\, dx$$

break up into 2 pieces:

$$\int \frac{3x}{x^2+1}\, dx + \int \frac{4}{x^2+1}\, dx$$

u-sub $u = x^2+1$

antideriv. is $4\tan^{-1} x$

do this by u-sub: $u = x-4$

$$\int \frac{7\, du}{u} = 7\ln(u)$$
$$= 7\ln(x-4)$$

$$\int \frac{8}{x^2+9}\, dx$$

$x^2+9 \rightsquigarrow x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{3}\right)$

$$= \int \frac{8}{9(\tan^2\theta+1)} 3\sec^2\theta\, d\theta = \frac{8}{9}\int \frac{3\sec^2\theta}{\sec^2\theta}\, d\theta = \frac{8}{3}\int d\theta = \frac{8}{3}(\theta) = \frac{8}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

[or just u-sub: $u = \frac{x}{3} \longrightarrow \int \frac{8 \cdot 3\, du}{9(u^2+1)} = \frac{8}{3}\int \frac{du}{u^2+1} = \frac{8}{3}\tan^{-1}(u) = \frac{8}{3}\tan^{-1}\left(\frac{x}{3}\right)$]

check: $\frac{d}{dx}\left(\frac{8}{3}\tan^{-1}\left(\frac{x}{3}\right)\right) = \frac{8}{3}\frac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\frac{1}{3}$

$$= \frac{8}{9}\cdot\frac{1}{1+\frac{1}{9}x^2} = \frac{8}{9+x^2} \quad ✓$$

Indeterminate forms

When we try to take a limit and get s.t. like

$0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, 1^∞ ...

call it <u>indeterminate form</u>, need some trick:

e.g. $\lim\limits_{x\to\infty} \frac{x^2-4}{3x^2+x-7}$ $\frac{\infty}{\infty}$ divide top, bottom by x^2

$$= \lim_{x\to\infty} \frac{1-\frac{4}{x^2}}{3+\frac{1}{x}-\frac{7}{x^2}} = \frac{1}{3}$$

e.g. $\lim\limits_{t\to 0} \frac{3e^t-3}{t}$ $\frac{0}{0}$ $= \lim\limits_{t\to 0} \frac{3e^t}{1} = \frac{3}{1} = \underline{\underline{3}}$ (L'H rule)

e.g. for 1^∞: like $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x$. Take log:

Write $L = \ln \lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim\limits_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x$

$$= \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) \to \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\left(\frac{1}{x}\right)} \to \frac{0}{0}$$

then use L'H: $= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\left(-\frac{2}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \underline{\underline{2}}$

So the answer to the original limit is $e^L = \underline{\underline{e^2}}$.

<u>Long division</u>: In partial frac. when num. is not lower degree than denominator

e.g. $$\int \frac{x^3+4x}{x^2-1}\,dx$$

$$\begin{array}{r} x \\ x^2+0x-1 \overline{) x^3+0x^2+4x} \\ \underline{x^3+0x^2-x} \\ 5x \end{array}$$

$$\frac{x^3+4x}{x^2-1} = x + \frac{5x}{x^2-1}$$

<u>Sequences</u>

$a_n = \frac{1}{n^2} + \frac{1}{n} + n^2 e^{-n}$: think about

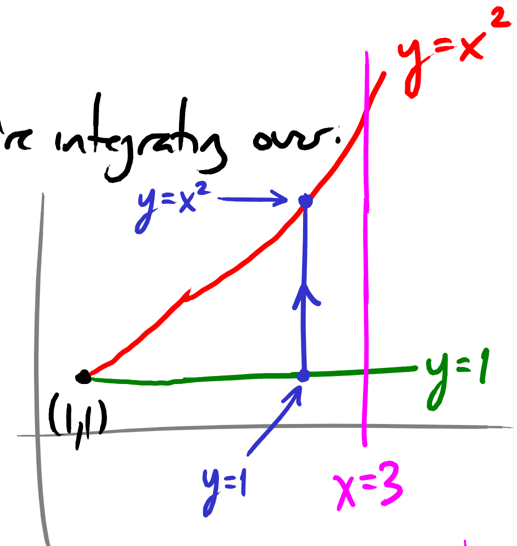
$$f(x) = \frac{1}{x^2} + \frac{1}{x} + x^2 e^{-x}$$

$\lim_{x\to\infty} f(x) = 0$ [using L'H rule!] <u>so</u> $\lim_{n\to\infty} a_n = 0$

Reversing order of integrals:

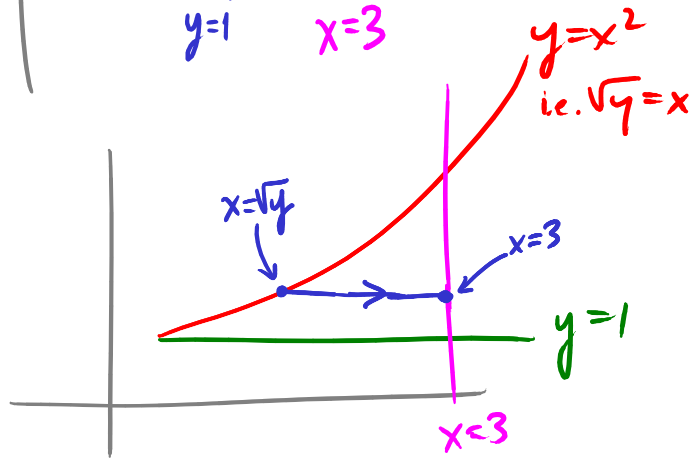
$$\int_1^3 \left[\int_1^{x^2} F(x,y)\, dy\right] dx$$

Draw a picture of the region you're integrating over:



Then draw the horizontal slicing:

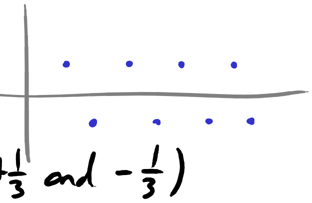
$$\int_1^9 \left[\int_{\sqrt{y}}^{3} F(x,y)\, dx\right]$$



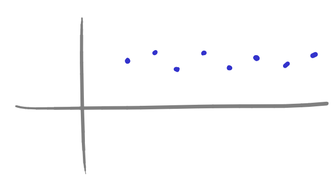
A couple more sequences:

$$a_n = (-1)^n \frac{n^2+1}{3n^2+2} = (-1)^n \cdot \frac{1+\frac{1}{n^2}}{3+\frac{2}{n^2}} \sim (-1)^n \cdot \frac{1}{3}$$ diverges

(oscillates between $+\frac{1}{3}$ and $-\frac{1}{3}$)



$$a_n = \frac{3n^2+(-1)^n n}{7n^2+\cos n} = \frac{3+\frac{1}{n}\cdot(-1)^n}{7+\frac{1}{n^2}\cos(n)}$$ converges to $\frac{3}{7}$



Lecture 31 — 7 Apr 2010

Last time: sequences. a_n

Recall summation notation e.g. $1+4+9+16+25$
$$= 1^2+2^2+3^2+4^2+5^2$$
$$= \sum_{i=1}^{5} i^2$$

Series (or Infinite Series) (Ch 12.2)

Take a sequence a_n. Try to take sum of all the terms of the seq.

Ex $a_i = i$: $1+2+3+4+5+\cdots = \sum_{i=1}^{\infty} i$

Ex $a_i = \frac{1}{2^i}$: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = \sum_{i=1}^{\infty} \frac{1}{2^i}$

What do these sums mean?
Like an improper integral $\int^{\infty}$: look at the sum of n terms,

$$S_n = \sum_{i=1}^{n} a_i \qquad (\text{"partial sum"})$$

Then try to take the limit as $n \to \infty$. If it exists, we say the series converges, otherwise it diverges.

Ex if $a_i = i$:
$$s_1 = 1$$
$$s_2 = 1+2 = 3$$
$$s_3 = 1+2+3 = 6$$
$$s_4 = 1+2+3+4 = 10$$

The s_n <u>don't</u> converge (i.e. $\lim\limits_{n\to\infty} s_n$ doesn't exist)

So $\sum\limits_{i=1}^{\infty} i$ doesn't converge (diverges).

<u>Ex</u> if $a_i = \frac{1}{2^i}$:

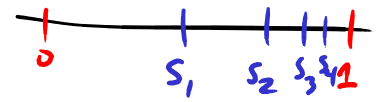
$$s_1 = \frac{1}{2}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\vdots$$



$\lim\limits_{n\to\infty} s_n = 1$. So we say $\sum\limits_{i=1}^{\infty} \frac{1}{2^i} = 1$ (<u>converges</u> to 1).

[<u>NB</u>: $s_n - s_{n-1} = a_n$]

Basic example: <u>geometric series</u> e.g. $2, 6, 18, 54, \ldots$ $\begin{bmatrix} a=2 \\ r=3 \end{bmatrix}$

$\times 3 \quad \times 3 \quad \times 3$

$a_n = a \cdot r^{n-1}$

1st term ratio of successive terms

What's $\sum_{i=1}^{\infty} a \cdot r^{i-1}$?

Look at the partial sums:

$$s_n = \sum_{i=1}^{n} a \cdot r^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$$

subtract 2 equations:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ - \quad r s_n &= \phantom{a + {}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ \hline s_n - r s_n &= a - ar^n \\ s_n(1-r) &= a(1-r^n) \end{aligned}$$

So: $s_n = a \dfrac{1-r^n}{1-r}$ (if $r \neq 1$)

We're interested in $\lim_{n\to\infty} s_n$. If $|r|<1$ then $\lim_{n\to\infty} r^n = 0$

so $\lim_{n\to\infty} s_n = \dfrac{a}{1-r}$

If $|r|>1$ or $r=-1$ then $\lim_{n\to\infty} s_n$ doesn't exist (b/c $\lim_{n\to\infty} r^n$ doesn't)

Also if $r=1$ $\lim_{n\to\infty}$ doesn't exist

Summary: $$\sum_{i=1}^{\infty} a \cdot r^{i-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

Ex Find the sum of the series

$$2+\frac{1}{3}+\frac{1}{18}+\frac{1}{108}+\frac{1}{648}+\cdots$$

$\times\frac{1}{6}$ $\times\frac{1}{6}$ $\times\frac{1}{6}$ $\times\frac{1}{6}$

Geometric: $a=2$, $r=\frac{1}{6}$

$$\text{sum}=\frac{a}{1-r}=\frac{2}{1-\frac{1}{6}}=\frac{2}{\left(\frac{5}{6}\right)}=\frac{12}{5}$$

Ex $4-3+\frac{9}{4}-\frac{27}{16}+\cdots$

$\times-\frac{3}{4}$ $\times-\frac{3}{4}$ $\times-\frac{3}{4}$

Geometric: $a=4$, $r=-\frac{3}{4}$

$\left|-\frac{3}{4}\right|<1 \Rightarrow$ converges

$$\text{sum}=\frac{a}{1-r}=\frac{4}{1-\left(-\frac{3}{4}\right)}=\frac{4}{\left(\frac{7}{4}\right)}=\frac{16}{7}$$

Ex Does $\sum_{i=1}^{\infty}\frac{10^i}{(-9)^{i-1}}$ converge?

First: is it geometric? Yes. 2 ways to see this:

METHOD 1

$$10\cdot\frac{10^{i-1}}{(-9)^{i-1}}=10\cdot\left(-\frac{10}{9}\right)^{i-1}=ar^{i-1}$$

so our series is $\sum_{i=1}^{\infty}ar^{i-1}$ with $a=10$ and $r=-\frac{10}{9}$

METHOD 2

Divide $\dfrac{a_{i+1}}{a_i} = \dfrac{10^{i+1}/(-9)^i}{10^i/(-9)^{i-1}} = \dfrac{10}{-9}$

Since this ratio is constant, we have a geometric series, with $r = -\frac{10}{9}$

and $a = a_1 = \dfrac{10^1}{(-9)^{1-1}} = 10$

Since the series is geometric, with $|r| = \left|-\frac{10}{9}\right| = \frac{10}{9} > 1$,

the series diverges.

Ex Calculate the sum $\displaystyle\sum_{i=1}^{\infty} \frac{3+5^i}{7^i}$.

This is not geometric, but it is the sum of two geometric series:

$$\sum_{i=1}^{\infty} \frac{3+5^i}{7^i} = \sum_{i=1}^{\infty} \frac{3}{7^i} + \sum_{i=1}^{\infty} \left(\frac{5}{7}\right)^i$$

geom. series with $a = \frac{3}{7}$, $r = \frac{1}{7}$

$\Rightarrow$ converges to $\dfrac{a}{1-r} = \dfrac{\frac{3}{7}}{1-\frac{1}{7}} = \dfrac{1}{2}$

geom. with $a = \frac{5}{7}$, $r = \frac{5}{7}$

$\Rightarrow$ converges to $\dfrac{\frac{5}{7}}{1-\frac{5}{7}} = \dfrac{5}{2}$

$$= \frac{1}{2} + \frac{5}{2} = 3$$

Lecture 32

9 Apr 2010

Last time: <u>series</u> $a_1 + a_2 + a_3 + \cdots$

$$= \sum_{i=1}^{\infty} a_i$$

[Can also have seq. beginning from $i=0$ instead of $i=1$
Then have series like $a_0 + a_1 + a_2 + \cdots$

$$= \sum_{i=0}^{\infty} a_i$$
]

<u>Partial sums</u> of $\sum_{i=1}^{\infty} a_i$ are $S_n = \sum_{i=1}^{n} a_i$

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$\vdots$

A common class of examples: <u>geometric series</u> $\sum_{i=1}^{\infty} ar^{i-1}$

$\left[\text{or } \sum_{i=0}^{\infty} ar^i\right]$

<u>Partial sums</u> of $\sum_{i=1}^{\infty} ar^{i-1}$ are $S_n = a \cdot \frac{1-r^n}{1-r}$

Taking $n \to \infty$ limit:

$$\sum_{i=1}^{\infty} ar^{i-1} \begin{cases} \underline{\text{converges}} \text{ to } \frac{a}{1-r} & \text{if } |r| < 1 \\ \underline{\text{diverges}} & \text{if } |r| \geq 1 \end{cases}$$

Ex Does $\sum_{i=0}^{\infty} \frac{\pi^i}{3^{i+1}}$ converge?

$$\sum_{i=0}^{\infty} \frac{\pi^i}{3^{i+1}} = \sum_{i=0}^{\infty} \frac{\pi^i}{3 \cdot 3^i} = \sum_{i=0}^{\infty} \frac{1}{3}\left(\frac{\pi}{3}\right)^i \quad \text{geometric: } a=\frac{1}{3},\ r=\frac{\pi}{3}$$

Since $|r| = \frac{\pi}{3} \approx \frac{3.1416}{3} > 1$, this geometric series **diverges**.

Ex Consider the repeating decimal

$$1.\overline{73} = 1.73737373\ldots$$

Write it as a fraction.

$$1.73737373\ldots$$

$$= 1 + 0.73 + 0.0073 + 0.000073 + \cdots$$

$$= 1 + \underbrace{\frac{73}{100} + \frac{73}{100^2} + \frac{73}{100^4} + \cdots}_{\text{geometric series with } a=\frac{73}{100},\ r=\frac{1}{100}} \qquad \left[\sum_{i=1}^{\infty} \frac{73}{100} \cdot \left(\frac{1}{100}\right)^{i-1}\right]$$

$$= 1 + \frac{a}{1-r} = 1 + \frac{73/100}{1 - 1/100} = 1 + \frac{73/100}{99/100} = 1 + \frac{73}{99} = \frac{172}{99}$$

Ex Consider the series

$$\sum_{i=0}^{\infty} \frac{(x+3)^i}{2^i}$$

For what values of x does it converge?

$$\sum_{i=0}^{\infty} \left(\frac{x+3}{2}\right)^i$$

geometric with $a=1$

$r = \dfrac{x+3}{2}$

The series will converge when $|r| < 1$.

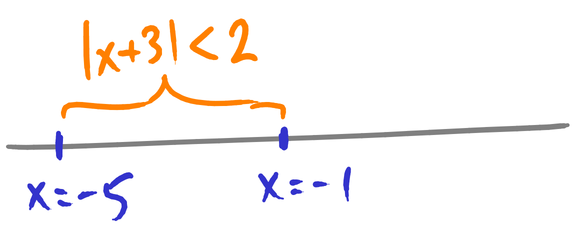
ie $\left|\dfrac{x+3}{2}\right| < 1$

$|x+3| < 2$

To find the boundaries of this region on the # line, solve $|x+3| = 2$

ie $x+3=2$ or $x+3=-2$

$x=-1$ or $x=-5$



So the series converges for $-5 < x < -1$.

Ex $\displaystyle\sum_{i=1}^{\infty} \frac{(\cos x)^i}{2^i}$

For what x does this series converge?

$$\sum_{i=1}^{\infty} \left(\frac{\cos x}{2}\right)^i \qquad \text{geometric:} \quad a = \frac{\cos x}{2}$$

$$r = \frac{\cos x}{2}$$

$$\left[\sum_{i=1}^{\infty} \left(\frac{\cos x}{2}\right) \cdot \left(\frac{\cos x}{2}\right)^{i-1}\right]$$

Converges if $|r| < 1$ i.e. $\left|\frac{\cos x}{2}\right| < 1$

i.e. $|\cos x| < 2$

That's true for all x: so this series converges for all x.

$\left[\text{to } \frac{a}{1-r}\right]$

Test For Divergence:

If $\lim_{i\to\infty} a_i$ doesn't exist, or if it exists but it's not 0,

then $\displaystyle\sum_{i=1}^{\infty} a_i$ diverges.

Ex $1+2+3+4+\cdots = \displaystyle\sum_{i=1}^{\infty} i$ diverges (b/c $\lim_{i\to\infty} i$ doesn't exist)

Ex $\displaystyle\sum_{i=1}^{\infty} \frac{3i+4}{4i-7}$ diverges (b/c $\displaystyle\lim_{i\to\infty} \frac{3i+4}{4i-7} = \frac{3}{4} \neq 0$)

Ex $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i}$: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ <u>might converge</u>

$\left(\lim_{i\to\infty} \frac{1}{i} = 0\right)$

– this test gives no info. here

Ex $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2}$: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$ <u>might converge</u>

$\left(\lim_{i\to\infty} \frac{1}{i^2} = 0\right)$

Ex $\displaystyle\sum_{i=0}^{\infty} \tan^{-1}(i)$: <u>diverges</u> $\left(\text{since } \lim_{i\to\infty} \tan^{-1}(i) = \frac{\pi}{2} \neq 0\right)$

In fact: $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i}$ <u>diverges</u>

$\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2}$ <u>converges</u> $\left(\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}\right)$

We'll see why, in the <u>next lecture</u>...

Lecture 33

12 Apr 2010

Last lecture: a few ways of seeing whether a series converges.

- If it's a geometric series, look at the common ratio r.
  - If $|r| \geq 1$, series diverges
  - If $|r| < 1$, series converges

How to tell whether the series is geometric?
Look at the ratios of successive terms: if they're all the same then the series is geometric.

Ex $2+4+6+8+10+\cdots$

$\frac{4}{2}=2 \quad \frac{6}{4}=\frac{3}{2} \quad \frac{8}{6}=\frac{4}{3} \quad \cdots$

these ratios aren't equal $\Rightarrow$ the series is not geometric.

Ex $1+2+4+8+16+\cdots$

$\frac{2}{1}=2 \quad \frac{4}{2}=2 \quad \frac{8}{4}=2 \quad \cdots$

the ratios are all $2 \Rightarrow$ the series is geometric, with $r=2$.

Since $2>1$, the series doesn't converge (diverges).

Another test (Test For Divergence):

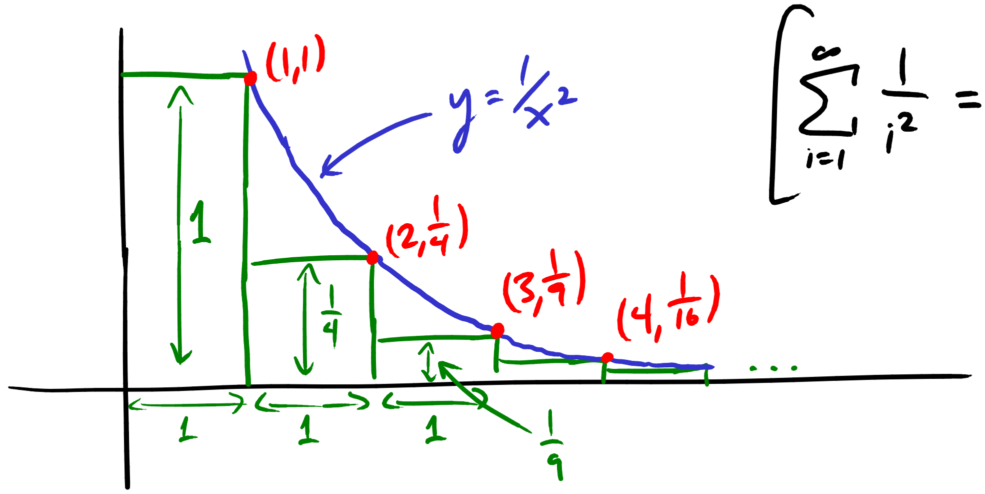
If $\lim_{i\to\infty} a_i$ doesn't exist, or if $\lim_{i\to\infty} a_i$ does exist but $\lim_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^{\infty} a_i$ diverges.

Integral Test (Ch 12.3)

Take the series $\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

Not geometric. And $\lim_{i \to \infty} \frac{1}{i^2} = 0$. So it *might* converge.

To see for sure:



$\left[\sum_{i=1}^{\infty} \frac{1}{i^2} = \text{sum of areas of these rectangles} \right]$

$A = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

The picture shows that $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ is <u>less than</u> $\int_1^{\infty} \frac{1}{x^2}\,dx$.

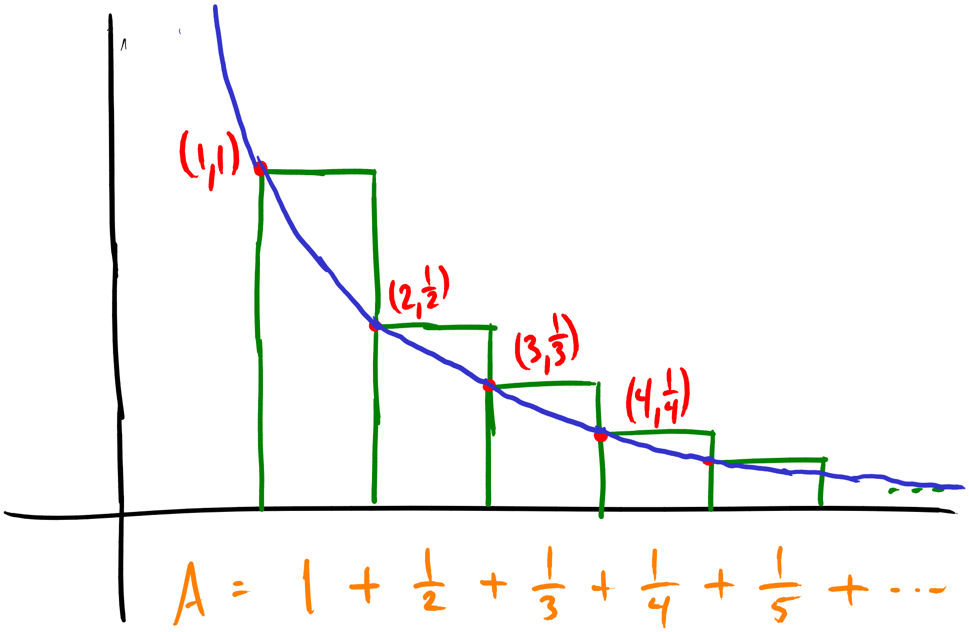
But $\int_1^{\infty} \frac{1}{x^2}\,dx$ <u>converges</u> (see lecture on improper integrals)

So $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ must <u>also converge</u>!

So: $\sum_{i=1}^{\infty} \frac{1}{i^2}$ <u>converges</u>.

(Adding the 1 in front to get $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ doesn't affect convergence.)

How about $\sum_{i=1}^{\infty} \frac{1}{i}$?



$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

The picture shows that $\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) > \int_1^\infty \frac{1}{x}\,dx.$

But $\int_1^\infty \frac{1}{x}\,dx$ diverges (cf. previous lecture on improper $\int$)

[because $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(t) = \infty$]

So $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges!

General rule (Integral Test):

Suppose $f(x)$ is a continuous decreasing function, defined for $1 \leq x < \infty$. Say $a_i = f(i)$.

Then:

If $\int_1^\infty f(x)\,dx$ is convergent then $\sum_{i=1}^{\infty} a_i$ is convergent

If $\int_1^\infty f(x)\,dx$ is divergent then $\sum_{i=1}^{\infty} a_i$ is divergent

Ex Does $\sum_{i=1}^{\infty} \frac{1}{i^2+1}$ converge or diverge?

$f(x) = \frac{1}{x^2+1}$ is <u>decreasing</u>

so look at $\int_1^{\infty} \frac{1}{x^2+1}\,dx$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \left(\tan^{-1} x \Big|_1^t \right) = \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4} \right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}, \text{ \underline{converges}}$$

So $\sum_{i=1}^{\infty} \frac{1}{i^2+1}$ <u>converges</u>, by the Integral Test.

Example For what values of p is $\sum_{i=1}^{\infty} \frac{1}{i^p}$ convergent?

("p-series")

If $p<0$: $\lim_{i\to\infty} \frac{1}{i^p} = \lim_{i\to\infty} i^{-p} = \infty$ so <u>diverges</u>

If $p=0$: $\lim_{i\to\infty} \frac{1}{i^p} = 1$ so <u>diverges</u>

If $p>0$: $f(x) = \frac{1}{x^p}$ is <u>decreasing</u> so use Integral Test:

look at $\int_1^\infty \frac{1}{x^p}\,dx$. We know this $\begin{cases} \underline{\text{converges}} \text{ if } p>1 \\ \underline{\text{diverges}} \text{ if } p\le 1 \end{cases}$

Summarizing (p-test):

$$\sum_{i=1}^{\infty} \frac{1}{i^p} \begin{cases} \underline{\text{converges}} \text{ if } p>1 \\ \underline{\text{diverges}} \text{ if } p\le 1 \end{cases}$$

Lecture 34 | 14 Apr 2010

Last time: "Integral Test":

If $f(x)$ is continuous, decreasing for $x > M$ $\quad (M = \text{any } \#)$

and $a_n = f(n)$

then $\displaystyle\sum_{n=M}^{\infty} a_n \begin{cases} \text{converges if } \int_M^\infty f(x)\,dx \text{ converges} \\ \text{diverges if } \int_M^\infty f(x)\,dx \text{ diverges} \end{cases}$

"p-test":

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Ex Does $\displaystyle\sum_{k=31}^{\infty} k e^{-k}$ converge?

Try to use Integral Test: look at $f(x) = xe^{-x}$

Is $f(x)$ decreasing? $\quad f'(x) = e^{-x} + x \cdot (-e^{-x})$

$$= \underbrace{(1-x)}_{\text{negative if } x>1} \underbrace{e^{-x}}_{\text{positive}}$$

So if $x > 1$, $f'(x) < 0$ i.e. $f(x)$ is decreasing if $x > 1$.

So we can apply Integral Test:

Look at $\displaystyle\int_1^\infty xe^{-x}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x}\,dx$

by Int. By Parts, can see that limit exists

so $\int_1^\infty x e^{-x}\, dx$ converges

so $\sum_{k=1}^{\infty} k e^{-k}$ converges by Int. Test

Comparison Tests (Ch 12.4)

Comparison Test:

Suppose we have two sequences of positive numbers: a_n, b_n

1) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$

then $\sum_{n=1}^{\infty} a_n$ is convergent.

2) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$

then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex $\sum_{n=1}^{\infty} \frac{5}{2n^3+4n+3}$: Say $a_n = \frac{5}{2n^3+4n+3}$, $b_n = \frac{5}{2n^3}$

$$\frac{5}{2n^3+4n+3} < \frac{5}{2n^3}$$

and $\sum_{n=1}^{\infty} \frac{5}{2n^3}$ converges by p-test $(p=3>1)$

So $\sum \frac{5}{2n^3+4n+3}$ converges by Comparison Test

Ex $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Comparison: $\frac{\ln n}{n} > \frac{1}{n}$ (when $n>3$)

And $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by p-test, with $p=1$)

So $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by Comparison Test.

Limit Comparison Test:

Suppose a_n, b_n are two sequences of positive numbers

and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $c \neq 0$

Then: if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

---

Ex Does $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converge? Say $a_n = \frac{1}{2^n-1}$, $b_n = \frac{1}{2^n}$

- $$\lim_{n\to\infty} \frac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-2^{-n}} = 1$$

- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$. Geometric series with $r = \frac{1}{2}$.
  Since $\left|\frac{1}{2}\right| < 1$, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

So: by Limit Comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

Ex Does $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$ converge?

For large n, $a_n = \frac{2n^2+3n}{\sqrt{n^5+5}} \sim \frac{2n^2}{\sqrt{n^5}} = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$

So let's try using Lim-Comp Test with $b_n = \frac{2}{n^{1/2}}$

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n^2+3n}{\sqrt{n^5+5}}\right)}{\left(\frac{2}{n^{1/2}}\right)} = \lim_{n\to\infty} \frac{\left[\frac{n^2(2+\frac{3}{n})}{n^{5/2}\sqrt{1+5n^{-5/2}}}\right]}{\left(\frac{2}{n^{1/2}}\right)}$

$$= \lim_{n\to\infty} \frac{2+\overset{\to 0}{\frac{3}{n}}}{2\sqrt{1+\underset{\to 0}{5n^{-5/2}}}} = \lim_{n\to\infty} \frac{2}{2} = 1$$

- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^{1/2}}$ diverges by p-test $(p = \frac{1}{2} < 1)$

So: by Lim-Comp Test, $\sum_{n=1}^{\infty} a_n$ diverges.

Ex Does $\sum_{n=1}^{\infty} \frac{n+1}{(n)4^n}$ converge?

Use Limit-Comparison Test with

$$a_n = \frac{n+1}{n4^n}, \quad b_n = \frac{1}{4^n}$$

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n4^n} \cdot 4^n = \lim_{n\to\infty} \frac{n+1}{n} = 1$
- And $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4^n}$ <u>converges</u> (geometric, $r = \frac{1}{4}$)

So: $\sum_{n=1}^{\infty} a_n$ <u>converges</u>.

Lecture 35 | 16 Apr 2010

What do we do with a series like:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

This is an example of an <u>alternating series</u>
(so called because the signs of the terms alternate: $+, -, +, -, +, -, \ldots$)
(could also have $-, +, -, +, -, +, \ldots$)

<u>Alternating Series Test</u>

Suppose we have the series $\sum_{n=1}^{\infty} (-1)^n b_n$ $\left[\text{or } \sum_{n=1}^{\infty} (-1)^{n+1} b_n\right]$
where all $b_n > 0$.

If the b_n are <u>decreasing</u> ($b_{n+1} \leq b_n$ for all n)
and $\lim_{n\to\infty} b_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^n b_n$ <u>converges</u>. $\left(\text{or } \sum_{n=1}^{\infty} (-1)^{n+1} b_n\right)$

<u>Ex</u> $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

This is alternating series with $b_n = \frac{1}{n}$.
Try applying Alt. Series Test:

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓
- Are b_n decreasing? $b_{n+1} = \frac{1}{n+1}$, $b_n = \frac{1}{n}$, and $\frac{1}{n+1} \leq \frac{1}{n}$

So $b_{n+1} \le b_n$

So Alt. Series Test says $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges.

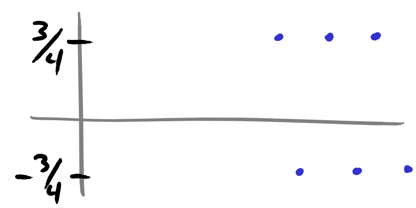
Ex $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1} = \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{3n}{4n-1}\right)$

alternating with $b_n = \frac{3n}{4n-1} > 0$

We could try Alt. Series Test:

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$ so the test doesn't apply.

In fact, write $a_n = (-1)^n \left(\frac{3n}{4n-1}\right)$, $\lim_{n\to\infty} a_n$ does not exist (oscillates)



So $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1}$ diverges by the "Test For Divergence"

Ex

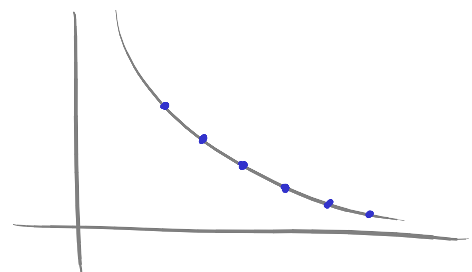
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$$

Alternating series: $b_n = \dfrac{n^2}{n^3+1}$

Use Alt. Series Test –

- $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{n^2}{n^3+1} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$ ✓

- Is $b_{n+1} \leq b_n$?



Look at $f(x) = \dfrac{x^2}{x^3+1}$

Take $f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2}$ (Quotient Rule + simplific.)

For large enough x,

positive negative

$$\frac{x(2-x^3)}{(x^3+1)^2}$$

positive

So $f'(x) < 0$ for large enough x

So b_n is decreasing ✓

So Alt. Series Test applies; so $\sum (-1)^{n+1} \dfrac{n^2}{n^3+1}$ converges.

__Ex__ $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}}$ $\cos(n\pi) = (-1)^n$!

So this is <u>really</u> $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$ $\longrightarrow$ Alternating, with $b_n = \frac{1}{n^{3/4}}$.

So can use Alt. Series Test.

Lecture 36

19 Apr 2010

Ex (from Lecture 34):

For what p does the series

$$\sum_{n=7}^{\infty} \left(\frac{n+1}{n^{5p}}\right) \cos(2\pi n)$$

converge?

First observation: $\cos(2\pi n) = 1$. So the series is really

$$\sum_{n=7}^{\infty} \frac{n+1}{n^{5p}}$$

Now at large n this would go $\sim \dfrac{n}{n^{5p}} = \dfrac{1}{n^{5p-1}}$

So try the Limit Comparison Test: using

$$a_n = \frac{n+1}{n^{5p}}, \quad b_n = \frac{1}{n^{5p-1}}$$

To see if the test applies:

calculate $\displaystyle \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n+1}{n^{5p}}\right)}{\left(\frac{1}{n^{5p-1}}\right)} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$

So the test applies: $\sum a_n$ converges if and only if $\sum b_n$ converges.

$\displaystyle \sum b_n = \sum \frac{1}{n^{5p-1}}$: use p-test — converges if $5p-1>1$

i.e $p > \frac{2}{5}$

So finally, $\sum a_n$ converges if $p > \frac{2}{5}$
diverges if $p \leq \frac{2}{5}$

Absolute Convergence

$\sum a_n$

Call $\sum a_n$ "absolutely convergent" if $\sum |a_n|$ is convergent.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots$ $\quad \left[a_n = \frac{(-1)^n}{n^2}\right]$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by p-test, $p = 2 > 1$)

So $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Fact: If $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.

If $\sum a_n$ is convergent but $\sum a_n$ is not absolutely convergent, then we call $\sum a_n$ conditionally convergent.

Ex $\sum (-1)^n \cdot \frac{1}{n}$ is convergent (by alt. series test)

But $\sum (-1)^n \cdot \frac{1}{n}$ is not absolutely convergent (because $\sum_{1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{1}^{\infty} \frac{1}{n}$ is divergent (by p-test))

So $\sum_{1}^{\infty} (-1)^n \cdot \frac{1}{n}$ is <u>conditionally convergent</u>

So have 3 possibilities:
- absolutely convergent
- conditionally convergent
- divergent

<u>Ex</u> $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Has both <u>positive</u> and <u>negative</u> terms:

$+, -, -, -, +, +, +, -, \ldots$

<u>Not</u> alternating.

<u>Is</u> it <u>absolutely</u> convergent? Look at $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n^2}\right|$

We know $\left|\frac{\cos(n)}{n^2}\right| = \frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}$

And we know $\sum_{1}^{\infty} \frac{1}{n^2}$ converges (p-test)

So $\sum_{1}^{\infty} \left|\frac{\cos(n)}{n^2}\right|$ converges by Comparison Test $\left[\begin{array}{l} a_n = \left|\frac{\cos(n)}{n^2}\right| \\ b_n = \frac{1}{n^2} \end{array}\right]$

So $\sum_{1}^{\infty} \frac{\cos(n)}{n^2}$ <u>converges absolutely</u>

$\left(\text{So } \sum \frac{\cos(n)}{n^2} \text{ converges}\right)$.

<u>Ex</u> $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ This is <u>alternating series</u> with $b_n = \dfrac{1}{\ln n}$.

So by alternating series test, it <u>converges</u>.

Does it converge <u>absolutely</u>?

i.e. does $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ <u>converge</u>?

$\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum \frac{1}{n}$ diverges —

so $\sum \frac{1}{\ln n}$ diverges by Comparison Test.

So $\sum (-1)^n \frac{1}{\ln n}$ <u>converges conditionally</u>.

Ratio Test

1) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$

then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ (or $= \infty$)

then $\sum_{n=1}^{\infty} a_n$ is divergent.

[If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ then the test is inconclusive.]

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$. Ratio test: $a_n = (-1)^n \frac{n^3}{3^n}$

$a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^{n+1}} \cdot \frac{(n+1)^3}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 = \frac{1}{3}$$

Since $L = \frac{1}{3} < 1$, $\sum a_n$ converges absolutely by Ratio Test.

Lecture 37 21 Apr 2010

Last time: Absolute and conditional convergence
Ratio Test

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\qquad a_n = \frac{n^n}{n!}$

Ratio Test: look at $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \frac{(n+1)^{n+1}}{n^n}\cdot\frac{n!}{(n+1)!}$$

[use $(n+1)! = (n+1)n!$]

$$= \frac{(n+1)\cdot(n+1)^n}{n^n}\cdot\frac{n!}{(n+1)n!}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

So $\displaystyle\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Since $e > 1$, Ratio Test says $\sum \frac{n^n}{n!}$ diverges.

Ex $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{1+n^2}$ $\qquad a_n = \frac{\sqrt{n}}{1+n^2}$

Suppose we try Ratio Test on this:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n+1}/1+(n+1)^2}{\sqrt{n}/1+n^2} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2}$$

and $\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2} = 1$ (skipped simplification steps here)

So the Ratio Test is <u>inconclusive</u> here.

(Could see that this $\sum$ converges using Limit-Comp Test, with $b_n = \frac{1}{n^{3/2}}$).

Root Test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$

  Then $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u>.

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ (or $= \infty$)

  Then $\sum_{n=1}^{\infty} a_n$ is <u>divergent</u>.

[If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then the Root Test is inconclusive.]

Ex $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ $\qquad a_n = \left(\frac{2n+3}{3n+2}\right)^n$

Root Test: $\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2}$

and $\lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$

so $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ converges (absolutely)

Strategy for Testing Series (Ch 12.7)

Classify the series according to its form.

1) $\sum \frac{1}{n^p}$: p-test.

2) $\sum ar^{n-1}$ or $\sum ar^n$: geometric series: converges if $|r|<1$, diverges if $|r| \geq 1$

3) If the series looks similar to a p-series or geom series: try comparison or limit-comparison (picking b_n to be the p-series or geometric series). (If the series has some negative terms then apply this method instead to $\sum |a_n|$ — i.e. test for absolute convergence.)

4) If you can see easily that $\lim_{n\to\infty} a_n \neq 0$, use Test For Divergence.

5) If the series is of the form $\sum_1^{\infty} (-1)^n b_n$ or $\sum_1^{\infty} (-1)^{n+1} b_n$

try Alternating Series Test.

6) If the series involves factorials (or other products involving n terms, including k^n) — try Ratio Test.

[But not for series where a_n is just rational function — Ratio Test will be inconclusive for those]

7) If $a_n = (\text{something})^n$ try Root Test.

8) If $a_n = f(n)$ and you know to do $\int_1^{\infty} f(x)\,dx$

[and $f(x)$ is decreasing for large enough x]

try Integral Test.

Ex $\sum_1^{\infty} \left(\frac{n^2+4}{3n^2+7n}\right)^{3n}$

$a_n = \left(\frac{n^2+4}{3n^2+7n}\right)^{3n}$ — use Root Test — $\sqrt[n]{\left(\frac{n^2+4}{3n^2+7n}\right)^{3n}}$

$$= \left(\frac{n^2+4}{3n^2+7n}\right)^3$$

...

Ex $\sum \frac{n+8}{2n+1}$: use Test For Divergence

Ex $\sum n^2 e^{-n^3}$: use Integral Test with $f(x) = x^2 e^{-x^3}$

Ex $\sum (-1)^n \frac{n^3}{n^4+1}$: use Alternating Series Test

[and if we want to see whether it's <u>absolutely</u> convergent, use Lim-Comp with $b_n = \frac{1}{n}$]

Ex $\sum \frac{2^k}{k!}$: use Ratio Test

Ex $\sum n \sin\left(\frac{1}{n}\right)$: use Test For Divergence

---

Ex $\sum \frac{1}{2+3^n}$: use Comparison or Lim-Comp with $b_n = \frac{1}{3^n}$

Ex $\sum (-1)^j \frac{\sqrt{j}}{j+5}$: use Alt. Series Test

Lecture 38 | 23 Apr 2010

Housekeeping:

Exam 3 Tue May 4 7-9pm WEL 2.246

covers Ch 12.1-12.11 NO CALCULATORS!

Strategy tip:

If the question is "Does this series $\sum a_n$
1) converge absolutely
2) converge conditionally
3) diverge"

you should check first whether $\sum |a_n|$ converges!
If it does, the answer is 1).

Power Series (Ch 12.8)

A power series centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

(a will be a constant, x is a variable)

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

is a power series centered at $a=0$.

Fact: For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are 3 possibilities:

Case 1) Series converges only when $x=a$.

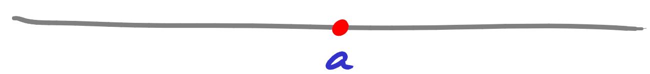
Case 2) Series converges for all values of x.

Case 3) There is some number $R>0$ such that
the series converges for $|x-a|<R$
the series diverges for $|x-a|>R$

We call R the "radius of convergence" of the series.
In case 1, we say $R=0$.
In case 2, we say $R=\infty$.

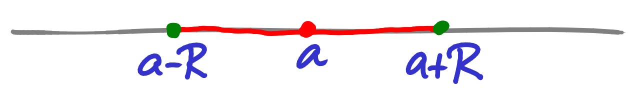
The "interval of convergence" is the set of all x where the series converges.

In case 1, it is just the point $x=a$.



In case 2, it is the whole line $-\infty<x<\infty$
or $x\in(-\infty,\infty)$.

In case 3, there are 4 possibilities for the interval:



$(a-R,\ a+R)$
$[a-R,\ a+R)$
$(a-R,\ a+R]$
$[a-R,\ a+R]$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = (x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} + \cdots$

Power series centered at $a=3$.

What is the interval of convergence?

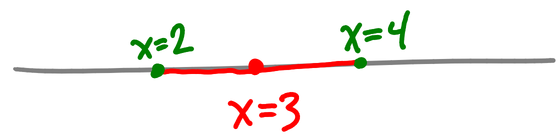
Use Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|^{n+1}/n+1}{|x-3|^n/n} = |x-3| \cdot \frac{n}{n+1} \longrightarrow |x-3|$$

as $n \to \infty$

So series converges if $|x-3| < 1$

diverges if $|x-3| > 1$



So the radius of convergence $R=1$

Does the series converge at the endpoints $x=2$ or $x=4$?

Ratio test is inconclusive there $(L=1)$

$x=4$: series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ divergent (p-test)

$x=2$: series is $\displaystyle\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergent (alt. series test)

Now we're done: interval of convergence is $[2,4)$

i.e. $2 \leq x < 4$

Ex $\displaystyle\sum_{n=1}^{\infty} n!\,x^n$ Power series centered at $a=0$.

Ratio Test: $\displaystyle\frac{|a_{n+1}|}{|a_n|} = \frac{|(n+1)!\,x^{n+1}|}{|n!\,x^n|} = \frac{|(n+1)\,n!\,x^{n+1}|}{|n!\,x^n|}$

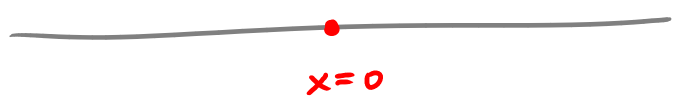
$$= (n+1)\frac{n!}{n!}\frac{|x|^{n+1}}{|x|^n} = (n+1)|x|$$

As $n\to\infty$, $\displaystyle\lim_{n\to\infty}(n+1)|x| = \begin{cases} \displaystyle\lim_{n\to\infty} 0 & \text{if } x=0 \\ \displaystyle\lim_{n\to\infty}(n+1)|x| & \text{if } x\neq 0 \end{cases}$

$$= \begin{cases} 0 & \text{if } x=0 \\ \infty & \text{for all } x\neq 0 \end{cases}$$

So the power series converges for $x=0$, diverges for all $x\neq 0$.

Radius of convergence $R=0$.



Interval of convergence is just a single point, $x=0$. (Case 1)

Ex $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ ("Bessel function")

Ratio test: $\displaystyle\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+2}}{2^{2n+2}((n+1)!)^2}\cdot\frac{2^{2n}(n!)^2}{|x|^{2n}}$

$$= \frac{|x|^2}{4}\cdot\left[\frac{n!}{(n+1)!}\right]^2 = \frac{|x|^2}{4}\cdot\left(\frac{1}{n+1}\right)^2 \to 0 \text{ as } n\to\infty$$

$L = 0 < 1$, so series converges — <u>for all</u> x!

Radius of convergence is $R = \infty$

Interval of convergence is $(-\infty, \infty)$.

<u>Ex</u> $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ Power series centered at $a=-2$.

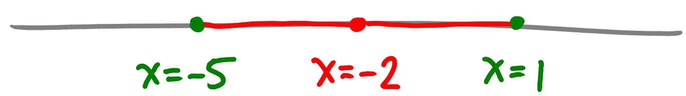
Ratio test: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\,|x+2|^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\,|x+2|^n}$

$$= \left(\frac{n+1}{3n}\right)|x+2| \longrightarrow \frac{1}{3}|x+2| \quad \text{as } n\to\infty$$

So series <u>converges</u> if $\frac{1}{3}|x+2| < 1$, i.e. $|x+2| < 3$

<u>diverges</u> if $|x+2| > 3$

Radius of convergence $R=3$



Does it converge at the ends $x=-5$ and $x=1$?

$x=1$: $\displaystyle\sum \frac{n\,3^n}{3^{n+1}} = \sum \frac{n}{3}$ diverges (Test For Div.)

$x=-5$: $\displaystyle\sum \frac{n(-3)^n}{3^{n+1}} = \sum \frac{n}{3}(-1)^n$ diverges (Test For Div.)

So the interval of convergence is $\underline{\underline{(-5, 1)}}$

Lecture 39

26 Apr 2010

Housekeeping:

- UT Learning Center exam review tonight and tomorrow 6-8pm
- Course Instructor surveys now open at
  https://utdirect.utexas.edu/diia/ecis/
- Exam 3 next Tuesday 7-9pm

A quick comment about sequences vs series:

Take the sequence $a_n = \frac{1}{n}$

We may ask 2 different questions about this sequence.

Q1) Does the sequence $\{a_n\}$ converge?

A1) Yes: $\lim\limits_{n\to\infty} \frac{1}{n} = 0.$

Q2) Does the series $\sum\limits_{n=1}^{\infty} a_n$ converge?

A2) No: $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test.

Last time: Power series $\sum\limits_{n=0}^{\infty} c_n (x-a)^n$

Functions As Power Series (Ch 12.9)

Remember the formula $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ (geometric series first term $= 1$ common ratio $= x$)

(for $|x| < 1$)

Ex: $\dfrac{1}{0.7} = \dfrac{1}{1-0.3} = \sum_{n=0}^{\infty} (0.3)^n$

$$= 1 + 0.3 + (0.3)^2 + (0.3)^3 + \cdots$$

$$= 1 + 0.3 + 0.09 + \cdots$$

$$\approx 1.4$$

Ex Find a representation of the function $\dfrac{1}{1+x^2}$ as a power series, and its interval of convergence, radius of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

To find interval of convergence: could use Ratio Test.
But here there's shortcut: this is a geometric series with common ratio $r = -x^2$.

So it converges if and only if $|r| < 1$

i.e. $|-x^2| < 1$

i.e. $|x|^2 < 1$

$|x| < 1$

So the interval of convergence is $\underline{(-1,1)}$. Radius of conv. $\underline{R=1}$.

Ex Write $\frac{1}{x+7}$ as a power series.

$$\frac{1}{x+7} = \frac{1}{7}\left(\frac{1}{\frac{x}{7}+1}\right)$$

$$= \frac{1}{7}\left(\frac{1}{1-\left(-\frac{x}{7}\right)}\right)$$

$$= \frac{1}{7}\left(\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n\right)$$

$$= \frac{1}{7}\sum_{n=0}^{\infty}\frac{(-1)^n}{7^n}x^n$$

$$= \underline{\underline{\sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}x^n}}$$ [power series centered at 0]

[We might also have written $\frac{1}{x+7} = \frac{1}{1-(-6-x)}$

and then get $\frac{1}{x+7} = \sum_{n=0}^{\infty}(-6-x)^n = \sum_{n=0}^{\infty}(-1)^n(x+6)^n$

That's another power series for the same function, centered at -6.]

Ex Write $\frac{x^4}{x+7}$ as a power series (centered at 0).

$$\frac{x^4}{x+7} = x^4\cdot\frac{1}{x+7} = x^4\cdot\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}x^n}$$

from the previous example

$$= \underline{\underline{\sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}x^{n+4}}}$$

Could also rewrite this: let $m = n+4$ (so $n = m-4$)

then the above also equals

$$= \sum_{m=4}^{\infty} \frac{(-1)^{m-4}}{7^{m-3}} x^m$$

$$= \sum_{m=4}^{\infty} \frac{(-1)^{m}}{7^{m-3}} x^m \qquad (-1)^{m-4} = (-1)^m(-1)^{-4}$$

<u>Fact</u>: If we have a power series for $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

Then also:

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}$$

$$\int f(x)\, dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n\, dx = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \right) + C$$

Both of these new series have the <u>same</u> radius of convergence as the original one.

<u>Ex</u> Express $\dfrac{1}{(1-x)^2}$ as a power series, find its radius of conv. (centered at $x=0$)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (\text{radius of conv} = 1)$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$ (also with radius of conv $= 1$)

<u>Ex</u> Express $\ln(1-x)$ as a power series (centered at 0).

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx$$

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} - C$$

To determine C: plug in $x=0$, then the eq. becomes

$$\ln(1) = 0 - C$$

$$0 = 0 - C \qquad \text{so} \qquad C = 0$$

so we get

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

[Also can rewrite this: set $m = n+1$, then

$$\ln(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}$$]

Lecture 40

28 Apr 2010

Last time: functions as power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (*)$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad (**)$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$ by applying $\frac{d}{dx}$ to both sides of $(*)$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$ by integrating both sides of $(*)$ with respect to x, and multiplying by -1

Could also similarly get

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$ by integrating both sides of $(**)$

How do we get power series representing a more general function $f(x)$?

Taylor (and Maclaurin) Series (Ch 12.10)

If we have any function f (which is "nice enough" — can be differentiated arbitrarily many times) and any number a, we can write down the Taylor series of f centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

[power series centered at a]

$\boxed{0!=1} \longrightarrow \quad = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

If this series has radius of convergence $R>0$ then its sum is $f(x)$ for $x \in (a-R, a+R)$. i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{for } |x-a|<R$$

If $a=0$ then we call this series the Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \text{for } |x|<R$$

(So "Maclaurin Series" = "Taylor series centered at $a=0$".)

Ex. Find the Maclaurin Series for $f(x)=e^x$ and its radius of convergence.

Macl. Series: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$\begin{aligned} f(x) &= e^x & f(0) &= e^0 = 1 \\ f^{(1)} = f'(x) &= e^x & f'(0) &= 1 \\ f^{(2)} = f''(x) &= e^x & f''(0) &= 1 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1 \end{aligned}$$

So the Macl. Series for $f(x)=e^x$ is just $\underline{\underline{\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n}}$

Radius of convergence: use Ratio Test $\quad a_n = \frac{1}{n!}x^n$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{(n)!}{|x|^n} = |x| \frac{n!}{(n+1)!} = \frac{|x|}{n+1}$$

[using $(n+1)! = (n+1)n!$]

$$\text{So } \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

Since $0 < 1$, Ratio Test says this series converges — for all values of x. So radius of convergence $R = \infty$.

[So $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all x.]

Ex Find the Maclaurin series for $f(x) = \sin x$ and its radius of convergence.

$f(x) = \sin x$	$f(0) = \sin(0) = 0$
$f'(x) = \cos x$	$f'(0) = \cos(0) = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin(0) = 0$
$f^{(3)}(x) = -\cos x$	$f^{(3)}(0) = -\cos(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = \sin(0) = 0$
$\vdots$	$\vdots$
(repeats with period 4)	(repeats with period 4)

Maclaurin series

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6 + \frac{f^{(7)}(0)}{7!}x^7 + \cdots$$

$$= 0 + \frac{1}{1!}\cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Maclaurin series for sin(x)

What is its radius of convergence?

Like in previous example, apply ratio test: you'll get $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$ for all x.

So $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.

Ex Find the first 3 terms of the Taylor series for $f(x) = \frac{1}{\sqrt{x}}$ centered at $a = 9$.

$f(x) = x^{-1/2}$ $\qquad f(9) = \frac{1}{3}$

$f'(x) = -\frac{1}{2}x^{-3/2}$ $\qquad f'(9) = -\frac{1}{54}$

$f''(x) = \frac{3}{4}x^{-5/2}$ $\qquad f''(9) = \frac{1}{324}$

First 3 terms of Taylor series:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \qquad a = 9$$

$$= \frac{1}{3} + \left(-\frac{1}{54}\right)(x-9) + \frac{1}{324\cdot 2}(x-9)^2$$

$$= \frac{1}{3} - \frac{1}{54}(x-9) + \frac{1}{648}(x-9)^2$$

[This is also called the "Taylor polynomial of degree 2 centered at $a=9$".]

Ex Find the Maclaurin series for $f(x) = x^3 \sin(x)$.

We could use the general formula for Maclaurin series, but there's a trick: we already know the Maclaurin series for $\sin(x)$.

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

So $$x^3 \sin(x) = x^3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+1)!} \quad \text{for all } x.$$

Ex Find the Maclaurin series for $\cos(x)$.

$$\cos(x) = \frac{d}{dx}\sin(x).$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n+1)!}(-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(-1)^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Tests for Series

Does $\sum a_n$ converge?

[$\sum$ always means $\sum_{n=M}^{\infty}$ below]

1) "Test for Divergence": If $\lim_{n\to\infty} a_n$ doesn't exist or $\lim_{n\to\infty} a_n \neq 0$,
Then $\sum a_n$ diverges

2) Geometric Series Test: If $\sum a_n$ is a geometric series with common ratio r, Then $\sum a_n$ $\begin{cases} \text{converges} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r|\geq 1 \end{cases}$

3) Integral Test. If $f(x)$ is a decreasing positive function and $a_n = f(n)$,
Then $\sum a_n$ and $\int_M^\infty f(x)\,dx$ either both converge or both diverge.

4) p-test. If $a_n = \frac{1}{n^p}$, Then $\sum a_n$ $\begin{cases} \text{converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p\leq 1 \end{cases}$

5) Comparison Tests.

a) If $a_n, b_n \geq 0$, $a_n \geq b_n$, and $\sum b_n$ diverges, Then $\sum a_n$ diverges.

b) If $a_n, b_n \geq 0$, $a_n \leq b_n$, and $\sum b_n$ converges, Then $\sum a_n$ converges.

6) Limit Comparison Test. If $a_n, b_n \geq 0$ and
$\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $c \neq 0$,
Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

7) Alternating Series Test.

If $\lim_{n\to\infty} b_n = 0$ and b_n are positive, decreasing $(b_{n+1} \le b_n)$

Then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

8) Ratio Test.

a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ Then $\sum a_n$ converges absolutely.

b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $= \infty$ Then $\sum a_n$ diverges.

9) Root Test.

a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ Then $\sum a_n$ converges absolutely.

b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $= \infty$ Then $\sum a_n$ diverges.

Lecture 41 30 Apr 2010

Housekeeping:

Exam 3 Tuesday, 7-9pm
- Ch 12.1-12.11
- 18 questions
- About as hard as Exam 2 (~)

Updated notes at http://www.ma.utexas.edu/users/neitzke/408L/
(Including a single PDF containing all lecture notes)

Uses of Taylor/Maclaurin Series (Ch 12.11)

Taylor series for $f(x)$ centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x) \qquad \text{for } |x-a|<R$$

We know a bunch of examples:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x|<1$$

$$\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n} \qquad |x|<1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{all } x$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{all } x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{all } x$$

Taylor polynomial of f, of degree d, centered at a:

$$T_d(x) = \sum_{n=0}^{d} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(first few terms of Taylor series: $\sum$ up to d instead of ∞)

Ex Use the Taylor polynomial of degree 2, centered at 0, for e^x to estimate $\sqrt[4]{e}$.

$f(x) = e^x$

Taylor poly: $T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

so $T_2(x) = 1 + x + \frac{x^2}{2}$

To get $\sqrt[4]{e} = e^{1/4}$: just plug in $x = \frac{1}{4}$

$$T_2\left(\tfrac{1}{4}\right) = 1 + \frac{1}{4} + \frac{1}{32} = \underline{\underline{\frac{41}{32}}}$$

$$\left[\begin{array}{l} \sqrt[4]{e} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad x = \frac{1}{4} \\ \text{we took } 1 + x + \frac{x^2}{2} \end{array}\right]$$

Ex Use the Taylor poly. of degree 3 for $\sin(x)$ centered at 0 to estimate $\sin\left(\frac{1}{10}\right)$.

We know Tay. series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The degree 3 Taylor poly. is

$$T_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

Plug in $x = \frac{1}{10}$:

$$T_3\left(\frac{1}{10}\right) = \frac{1}{10} - \frac{\left(\frac{1}{10}\right)^3}{6} = \frac{1}{10} - \frac{1}{6000} = \underline{\underline{\frac{599}{6000}}}$$

Ex Use the Taylor polynomial for e^{-x^2} of degree 2 centered at 0 to estimate

$$\int_{-1/2}^{1/2} e^{-x^2}\,dx.$$

Taylor series for e^{-x^2}:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Replace x by $-x^2$:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$$

To get Taylor poly, keep only terms involving x^d with $d \le 2$:

So $$T_2(x) = 1 - x^2$$

$$\int_{-1/2}^{1/2} T_2(x)\,dx$$

$$= \int_{-1/2}^{1/2} (1-x^2)\,dx = x - \tfrac{1}{3}x^3 \Big|_{-1/2}^{1/2}$$

$$= \underline{\underline{\frac{11}{12}}}$$

Another use of Taylor series:

<u>Ex</u> Calculate $\displaystyle\sum_{n=0}^{\infty} \left(-\frac{\pi^2}{16}\right)^n \frac{1}{(2n)!}$.

[Idea: this looks like the Taylor series for $\cos(x)$...]

$$\sum_{n=0}^{\infty} \left(-\frac{\pi^2}{16}\right)^n \frac{1}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n}}{(2n)!}$$

But $\displaystyle\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

So $$\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \underline{\underline{\frac{\sqrt{2}}{2}}}$$

Ex Find $\int_0^t \ln(1+x^3)\,dx$ as a power series.

Remember $\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$

$$\ln(1+x^3) = \sum_{n=1}^{\infty} -\frac{(-x^3)^n}{n}$$

$\ln(1+x^3) = \ln(1-(-x^3))$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}$$

So $$\int_0^t \ln(1+x^3)\,dx = \int_0^t \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}\,dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n+1}}{n(3n+1)} \Bigg|_{x=0}^{x=t}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \left[\frac{t^{3n+1}}{n(3n+1)} - \frac{0^{3n+1}}{n(3n+1)}\right]$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{3n+1}}{n(3n+1)}$$

Ex Find the Taylor polynomial of degree 1 for $\sqrt[3]{x}$ centered at $a=27$.

Taylor polynomial $T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\qquad f(27) = 3$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(27) = \frac{1}{3}(27)^{-2/3} = \frac{1}{3}\left(\frac{1}{9}\right) = \frac{1}{27}$

So $T_1(x) = 3 + \frac{1}{1!}\cdot\frac{1}{27}(x-27)$

$= \underline{\underline{3 + \frac{1}{27}(x-27)}}$

Lecture 42 | 3 May 2010

Exam 3 tomorrow 7–9pm NO CALCULATORS
WEL 2.246

18 questions:
all series
8 power series (incl. Taylor series, Taylor poly...)
7–8 tests for convergence of non-power series
2–3 other Q on non-power series

THINGS WORTH KNOWING:

Tests for convergence —

- Remember the "easy" examples

$$\sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} \qquad P, Q \text{ polynomials}$$

Can always do them by lim-comp and p-test.
The answer always just depends on the leading powers of n.

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{n^2-3n+9}{n^{5/2}+4}$ lim-comp to $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^{5/2}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$

which diverges by p-test ($p=\frac{1}{2}<1$).

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{n^4-7n}{n^6+8n^5}$ lim-comp to $\displaystyle\sum_{n=0}^{\infty} \frac{n^4}{n^6} = \sum \frac{1}{n^2}$

which converges by p-test $(p = 2 > 1)$.

- Always pay attention to what the Q asks:

  "Which of these series converges"

  "Which of these series diverges"

  "Which of them " converges conditionally"

- Most important tests are Ratio Test, p-test, Test For Divergence.

Test For Divergence: if $\lim_{n\to\infty} a_n \neq 0$ (or doesn't exist)

then $\sum_{n=0}^{\infty} a_n$ diverges.

Ex $\sum_{n=0}^{\infty} \frac{n}{\ln n}(-1)^n$ diverges by Test For Divergence

(in fact $\lim_{n\to\infty} \frac{n}{\ln n} = \infty$

so $\lim_{n\to\infty} \frac{n}{\ln n}(-1)^n$ doesn't exist)

- (But also remember other tests.)

- Root Test:

  Ex $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}+1}\right)^{3n}$

$$\sqrt[n]{\left|\frac{1}{\sqrt{n}+1}\right|^{3n}} = \left[\left(\frac{1}{\sqrt{n}+1}\right)^{3n}\right]^{1/n} = \left(\frac{1}{\sqrt{n}+1}\right)^{3n\cdot\frac{1}{n}}$$

$$= \left(\frac{1}{\sqrt{n}+1}\right)^{3} \xrightarrow[n\to\infty]{} \underset{L}{\overset{}{0}}$$

Since $L = 0 < 1$, the $\sum$ converges (absolutely).

[If we had gotten $L > 1$, then $\sum$ diverges.
If " " " $L = 1$, then we get no information (test inconclusive).]

Simplifying factorials in the Ratio Test:

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

Ratio Test: $$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{(n+1)!}{(2n+3)!}}{\dfrac{n!}{(2n+1)!}} = \frac{(2n+1)!}{(2n+3)!}\cdot\frac{(n+1)!}{n!}$$

Use $(n+1)! = (n+1)n!$
$(2n+3)! = (2n+3)(2n+2)(2n+1)!$

so the ratio simplifies to $$\frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!}\cdot\frac{(n+1)n!}{n!}$$

$$= \frac{n+1}{(2n+3)(2n+2)} \longrightarrow 0 \text{ as } n\to\infty$$

Since $0 < 1$, the series converges (absolutely).

- If Q asks whether a series is
    - absolutely conv.
    - conditionally conv.
    - divergent

check <u>absolute</u> <u>convergence</u> first!

<u>Ex</u> $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^3+4}$

Check absolute conv:

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{n+3}{n^3+4}\right| = \sum_{n=1}^{\infty} \frac{n+3}{n^3+4}$$

which can be done by lim-comp to $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$

which <u>converges</u> by p-test $(p=2)$

So $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^3+4}$ <u>converges absolutely</u>.

<u>Ex</u> $\sum \frac{3^n}{n^4+7n}$ <u>div</u> by Test For Div

$\sum \frac{n^2+6n}{3^n}$ <u>conv</u> by Ratio Test

$$\left[\frac{|a_{n+1}|}{|a_n|} = \frac{\left(\frac{(n+1)^2+6(n+1)}{3^{n+1}}\right)}{\frac{n^2+6n}{3^n}} \xrightarrow{n\to\infty} \frac{1}{3} < 1 \right]$$

• Remember that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

$\left(\text{also } \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{bn} = e^{ab}\right)$.

Power series:

Power series centered at a: $\sum_{n=0}^{\infty} c_n (x-a)^n$

Remember interval of convergence and radius of convergence:

Ex $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n n^3$ What are int. and rad. of conv.?

Use Ratio Test: $\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{x}{4}\right|^{n+1}(n+1)^3}{\left|\frac{x}{4}\right|^n (n^3)} = \left|\frac{x}{4}\right|\left(\frac{n+1}{n}\right)^3 \to \left|\frac{x}{4}\right|$ as $n\to\infty$

So the series converges if $\left|\frac{x}{4}\right| < 1$ i.e. $|x|<4$

diverges if $\left|\frac{x}{4}\right| > 1$ i.e. $|x|>4$

So radius of convergence $= 4$

Interval of convergence: is it $[-4,4]$ or $(-4,4]$

or $[-4,4)$ or $(-4,4)$?

Check endpoints: plug in $x=4$ $\sum_{n=0}^{\infty} (1)^n n^3$ diverges by TFD

plug in $x=-4$ $\sum_{n=0}^{\infty} (-1)^n n^3$ diverges by TFD

So int. of conv. is $(-4,4)$.

Remember the Taylor series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex Find Taylor series centered at $a=0$ for

$$x e^{2x^3}$$

Use the series for e^x: $\quad e^{2x^3} = \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!}$

$$x e^{2x^3} = x \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2^n x^{3n+1}}{n!}$$

Ex Find the Taylor polynomial of deg $=2$ around $a=4$ for $f(x)=\sqrt{x}$.

$$T_2(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} \quad \cdots$$

$f(x) = \sqrt{x}$ $\qquad\qquad$ $f(a) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad\qquad$ $f'(a) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f''(x) = -\frac{1}{4x^{3/2}}$ $\qquad f''(a) = -\frac{1}{4 \cdot 4^{3/2}} = -\frac{1}{32}$

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

Also remember <u>differentiating</u> and <u>integrating</u> series:

<u>Ex</u> Write $\int_0^t x \ln(1+x^2)\,dx$ as a power series.

<u>2 steps</u>:

First find a series for $x\ln(1+x^2)$:

$$x\ln(1+x^2) = x\ln(1-(-x^2)) = x\sum_{n=1}^{\infty} -\left(\frac{(-x^2)^n}{n}\right)$$

$$= \sum_{n=1}^{\infty} (-x)\frac{(-1)^n x^{2n}}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^{2n+1}}{n}$$

$$\int_0^t x\ln(1+x^2)\,dx = \int_0^t \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^{2n+1}}{n}\,dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^{2n+2}}{n(2n+2)}\Bigg|_{x=0}^{x=t}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\frac{t^{2n+2}}{2n+2}$$

Lecture 43

Final: 3 hours (7-10pm) next Thursday

25 problems (8 on seq/series)
(3 on multivariate)

Final Review

- If $f''(x) = \sqrt{x}$, $f(0) = 1$, $f'(0) = 2$

  What is $f(1)$?

$$f'(x) = \frac{2}{3}x^{3/2} + C$$

$$f(x) = \frac{4}{15}x^{5/2} + Cx + D$$

To get C and D: $f(0) = \frac{4}{5}(0)^{5/2} + C(0) + D = 1$

so $D = 1$

$f'(0) = \frac{2}{3}(0)^{3/2} + C = 2$

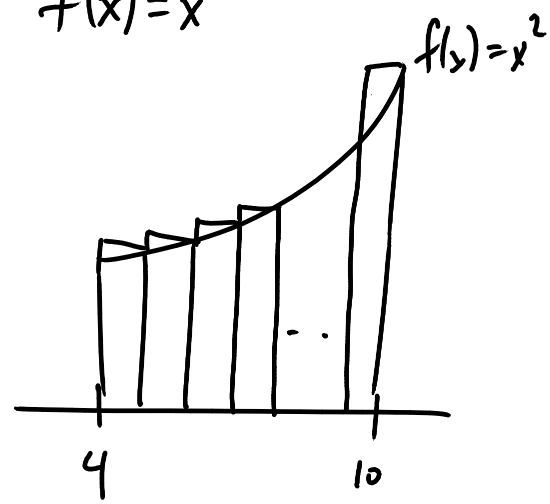
so $C = 2$

So $f(x) = \frac{4}{15}x^{5/2} + 2x + 1$

$$f(1) = \frac{4}{15} + 2 + 1 = \underline{\underline{\frac{49}{15}}}$$

- Write a formula for the area under the curve $f(x) = x^2$ on the interval $[4, 10]$.

(using right endpoints)



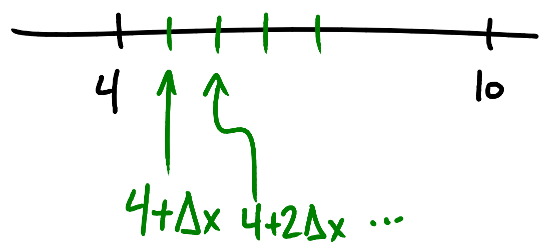
Chop $[4,10]$ into n intervals:

width $\Delta x = \frac{10-4}{n} = \frac{6}{n}$

right endpt of i^{th} interval $4 + i\Delta x = 4 + \frac{6i}{n} = x_i$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 + \frac{6i}{n}\right)^2 \frac{6}{n}$$



- If $g(x) = \int_{30}^{x^2} \tan u \, du$

what is $g'(x)$?

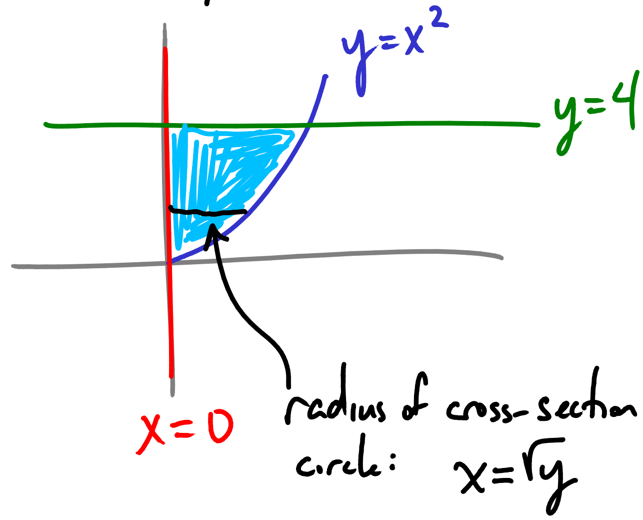
Use FTC: $\frac{d}{dx}\int_{30}^{x} f(u)\,du = f(x)$

and remember chain rule: $\frac{d}{dx} g(x) = 2x \tan x^2$

• Find the volume of the solid obtained by rotating the region bounded by

$y=x^2$
$x=0$
$y=4$

around the y-axis.



Chop the solid into cross-sections at fixed y.

Each x-section is a circle with radius $=\sqrt{y}$

So $V=\int_0^4 \pi r^2\,dy=\int_0^4 \pi y\,dy=\pi\frac{y^2}{2}\Big|_0^4=\underline{\underline{8\pi}}$

Methods of integration:

Remember $\int$ by parts

• Find $\int 6x(\ln x)^2\,dx$

IBP: $u=(\ln x)^2 \qquad v=3x^2$

$du=2\frac{\ln x}{x}dx \qquad dv=6x\,dx$

$uv-\int v\,du=(\ln x)^2 3x^2-\int 3x^2\cdot 2\frac{\ln x}{x}\,dx$

$$= (\ln x)^2 3x^2 - \int 6x \ln x \, dx$$

IBP again: $u = \ln x \quad v = 3x^2$

$du = \frac{1}{x} dx \quad dv = 6x \, dx$

$$= (\ln x)^2 3x^2 - \left(uv - \int v \, du\right)$$

$$= (\ln x)^2 3x^2 - 3x^2 \ln x + \int 3x^2 \frac{dx}{x}$$

$$= \underline{\underline{(\ln x)^2 3x^2 - 3x^2 \ln x + \frac{3}{2} x^2 + C}}$$

Find $\displaystyle\int_0^{\pi/3} \frac{\sec x \tan x}{5 - \sec x} dx$

Remember

$d(\sec x) = \sec x \tan x \, dx$

Substitute $u = 5 - \sec x$

$du = -\sec x \tan x \, dx$

$x = 0$ is $u = 5 - 1 = 4$

$x = \pi/3$ is $u = 5 - 2 = 3$

$$\int_4^3 \frac{-du}{u} = -\ln(u) \Big|_{u=4}^{u=3}$$

$$= -\left[\ln(3) - \ln(4)\right] = -\ln\left(\frac{3}{4}\right) = \underline{\underline{\ln\left(\frac{4}{3}\right)}}$$

Remember basic facts about integrals:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Partial fractions can always be used for $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

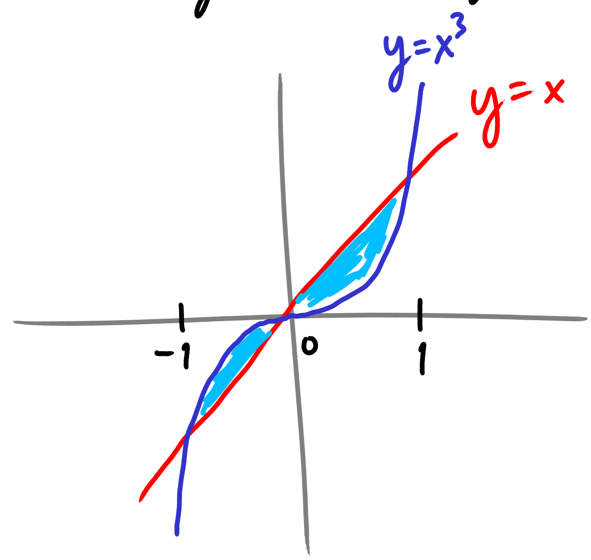
$$\left[\text{ex} \quad \int \frac{2x^2+3x+1}{x^2+x-2}\,dx \right]$$

Area between curves:

What's the area enclosed by the graphs of $y=x$ and $y=x^3$?

Solve $x=x^3$ to find the intersections:

$$x^3-x=0$$
$$x(x^2-1)=0$$
$$x(x+1)(x-1)=0$$



$$A = \int_{-1}^{0} (x^3-x)\,dx + \int_0^1 (x-x^3)\,dx$$

$$= \frac{1}{4} + \frac{1}{4} = \underline{\underline{\frac{1}{2}}}$$

Improper integrals

Just remember $\int_a^\infty f(x)\,dx$ means $\lim_{t\to\infty} \int_a^t f(x)\,dx$

Ex

$$\int_2^\infty \sin(x)\,dx = \lim_{t\to\infty} \int_2^t \sin(x)\,dx$$

$$= \lim_{t\to\infty} -\cos(x)\Big|_2^t$$

$$= \lim_{t\to\infty} \left(-\cos(t) + \cos(2)\right)$$

doesn't exist (so $\int$ diverges)

Ex

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \tan^{-1}(t) - \tan^{-1}(0)$$

$$= \frac{\pi}{2} - 0 = \underline{\underline{\frac{\pi}{2}}}$$