Lecture 16

24 Feb 2010

(More) trigonometric integrals (Ch 8.2)

$$\int \sin^5\theta \cos\theta \, d\theta$$

$u = \sin\theta$
$du = \cos\theta \, d\theta$

$$= \int u^5 \, du$$

Similarly for $\int \sin^a\theta \cos\theta \, d\theta$

or for $\int \cos^b\theta \sin\theta \, d\theta$

But what about e.g. $\int \sin^3\theta \, d\theta$?

Ex $\int \sin^3\theta \, d\theta = \int \sin^2\theta (\sin\theta \, d\theta)$

Use $\sin^2\theta + \cos^2\theta = 1$
$\sin^2\theta = 1 - \cos^2\theta$

So $\int = \int (1-\cos^2\theta)(\sin\theta \, d\theta)$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$
$-du = \sin\theta \, d\theta$

$$= \int (1-u^2)(-du)$$

$$= \int (u^2-1) \, du = \frac{u^3}{3} - u + C = \underline{\underline{\frac{1}{3}\cos^3\theta - \cos\theta + C}}$$

Ex

$$\int \sin^5\theta \cos^2\theta \, d\theta$$

$$= \int \sin^4\theta \cos^2\theta \, (\sin\theta \, d\theta)$$

Want $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$-du = \sin\theta \, d\theta$

$$= \int (\sin^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$$

$$= \int (1-\cos^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$$

$$= \int (1-u^2)^2 u^2 \, (-du)$$

$$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7}$$

$$= -\frac{\cos^3\theta}{3} + \frac{2\cos^5\theta}{5} - \frac{\cos^7\theta}{7}$$

General rule for $\int \sin^a\theta \cos^b\theta \, d\theta$:

If a odd, then pick off one of sines, write $(\sin\theta \, d\theta)$, use $\sin^2\theta = 1-\cos^2\theta$ to eliminate the rest of the sines, use $u = \cos\theta$.

If b odd, then pick off a cosine, write $(\cos\theta \, d\theta)$, use $\cos^2\theta = 1-\sin^2\theta$ to elim. rest of cosines, use $u = \sin\theta$.

What about <u>even</u> powers?

Half-angle formulas:

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

Ex $\int \sin^2\theta \, d\theta$

$$= \int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \int \frac{1}{2} - \frac{1}{2}\cos 2\theta \quad d\theta$$

$$= \frac{\theta}{2} - \frac{1}{2}\left(\frac{1}{2}\sin 2\theta\right) + C = \underline{\underline{\frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C}}$$

Ex $\int \cos^4\theta \, d\theta$

$$= \int (\cos^2\theta)^2 \, d\theta$$

$$= \int \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$

$$= \frac{1}{4}\int \left(1+2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right) d\theta$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \underline{\underline{\frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C}}$$

Ex $\int \tan^6 x \sec^4 x \, dx = ?$

Similar rules to what we used for sin, cos above:

$$\int = \int \tan^6 x \sec^2 x \left(\sec^2 x \, dx\right)$$

Want $u = \tan x$

$du = \sec^2 x \, dx$

Use $\sec^2 x = 1 + \tan^2 x$

$$= \int \tan^6 x \left(1+\tan^2 x\right) \left(\sec^2 x \, dx\right)$$

$$= \int u^6 \left(1+u^2\right) du$$

$$= \int \left(u^6 + u^8\right) du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \underline{\underline{\frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C}}$$

Same strategy whenever $\underline{\sec x}$ appears to an <u>even</u> power.

Ex $\displaystyle\int_0^{\pi/4} \tan^3 x \sec^5 x \, dx$

$$= \int_0^{\pi/4} \tan^2 x \sec^4 x \, (\tan x \sec x) \, dx$$

Want $u = \sec x$
$du = \sec x \tan x \, dx$

use $\tan^2 x = \sec^2 x - 1$

$$\int_0^{\pi/4} (\sec^2 x - 1) \sec^4 x \, (\tan x \sec x) \, dx$$

$$= \int_1^{\sqrt{2}} (u^2 - 1) u^4 \, du$$

$$= \cdots \qquad = \frac{2}{35}(1 + 6\sqrt{2})$$

Same strategy works for $\int \tan^a x \sec^b x \, dx$

whenever a is odd (and $b \geq 1$)

Handy facts:

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Ex

$$\int \tan^3 x \, dx$$

$$= \int \tan x \cdot \tan^2 x \, dx$$

$$= \int \tan x \left(\sec^2 x - 1\right) dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$\left[\begin{array}{l} u = \tan x \\ du = \sec^2 x \end{array} \Rightarrow \int u \, du = \frac{1}{2}u^2\right]$$

$$= \frac{1}{2}\left(\tan^2 x\right) - \ln\left|\sec x\right| + C$$

Ex

$$\int \sin 4x \cos 7x \, dx$$

Use product-to-sum identities:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$= \int \frac{1}{2}\left(\sin(-3x) + \sin(11x)\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{3}\cos(3x) - \frac{1}{11}\cos(11x)\right) + C$$