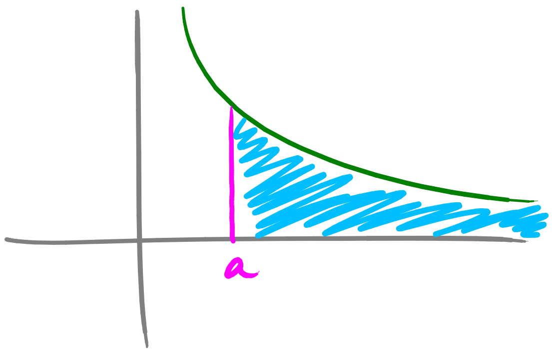
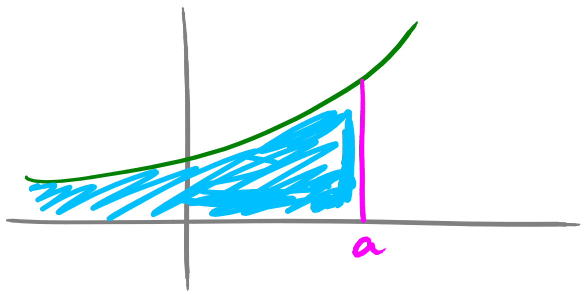
Lecture 23

12 Mar 2010

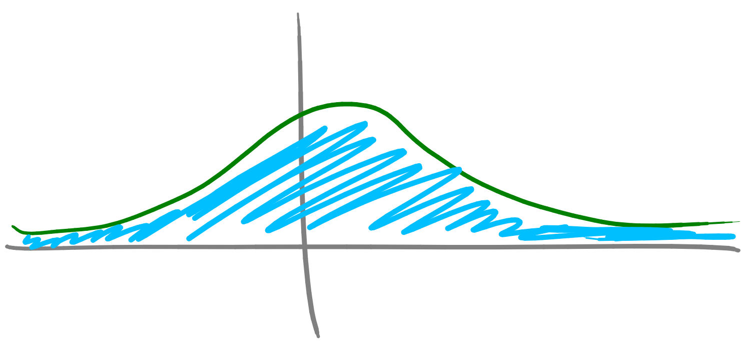
Last time: improper int.



$$\int_a^{\infty} f(x)\, dx$$



$$\int_{-\infty}^{a} f(x)\, dx$$



$$\int_{-\infty}^{\infty} f(x)\, dx$$

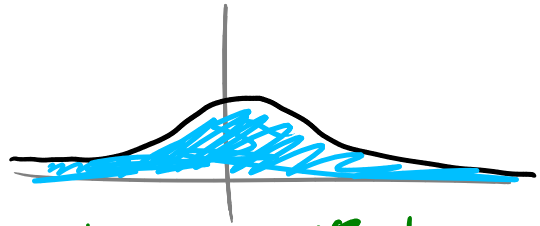
This is defined by splitting it up:

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_0^{\infty} f(x)\, dx$$

$$= \left(\lim_{t\to-\infty} \int_t^0 f(x)\, dx\right) + \left(\lim_{t\to\infty} \int_0^t f(x)\, dx\right)$$

[If either of these lim. does not exist, we say the integral is divergent; otherwise it's convergent]

Ex $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$



$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \qquad \left(\text{also } = 2\int_0^{\infty} \frac{1}{1+x^2}\,dx\right)$$

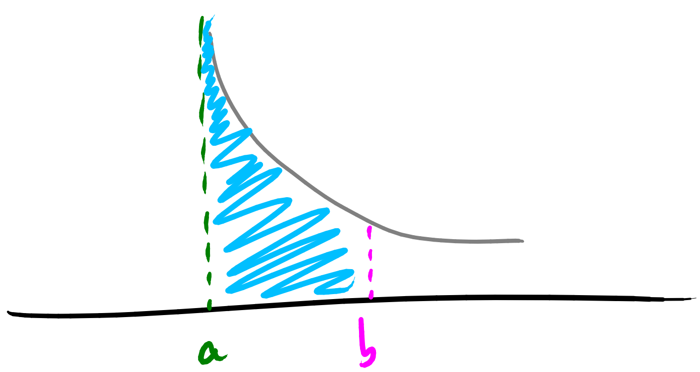
$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

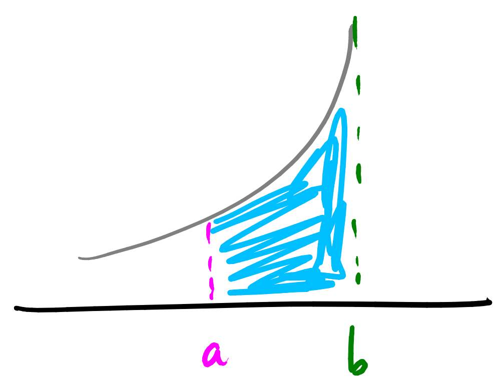
$$= \lim_{t\to-\infty} \tan^{-1}x \Big|_t^0 + \lim_{t\to\infty} \tan^{-1}x \Big|_0^t$$

$$= \lim_{t\to-\infty} -\tan^{-1}t + \lim_{t\to\infty} \tan^{-1}t$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \underline{\underline{\pi}}$$

<u>Another kind of imp. $\int$ we saw last time:</u>





Ex $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$: improper b/c $\dfrac{1}{\sqrt{x-2}}$ goes to ∞ as $x \to 2^+$.

So to define this int:

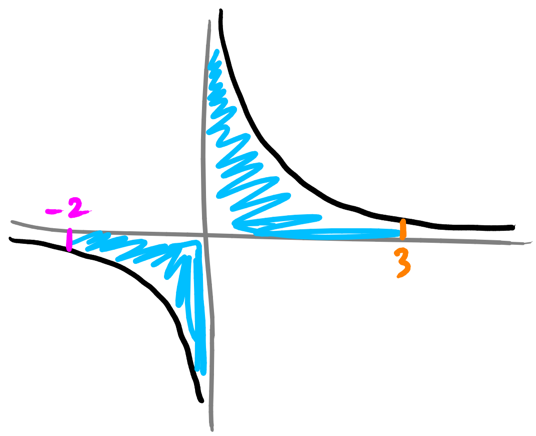
$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t\to 2^+} \left(2\left(\sqrt{3}-\sqrt{t-2}\right)\right)$$

(get this by u-sub: $u=x-2$)

$$= 2\sqrt{3} \quad \left(\underline{\text{convergent}}\right)$$

<u>Ex</u> $\displaystyle\int_{-2}^{3} \frac{1}{x}\,dx$



<u>Improper</u> b/c of vert. asymp. at $x=0$

$$\int_{-2}^{3} \frac{1}{x}\,dx = \int_{-2}^{0} \frac{1}{x}\,dx + \int_{0}^{3} \frac{1}{x}\,dx$$

$$= \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x}\,dx + \lim_{t\to 0^+} \int_{t}^{3} \frac{1}{x}\,dx$$

$$= \lim_{t\to 0^-} \left(\ln|x|\Big|_{-2}^{t}\right) + \lim_{t\to 0^+} \left(\ln x\Big|_{t}^{3}\right)$$

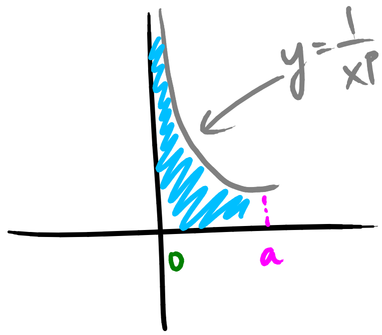
$$= \lim_{t\to 0^-} \underbrace{\left(\ln|t| - \ln 2\right)}_{\downarrow\, -\infty} + \lim_{t\to 0^+} \underbrace{\left(\ln 3 - \ln|t|\right)}_{\downarrow\, -\infty}$$

so <u>neither</u> of these limits goes to a finite #:

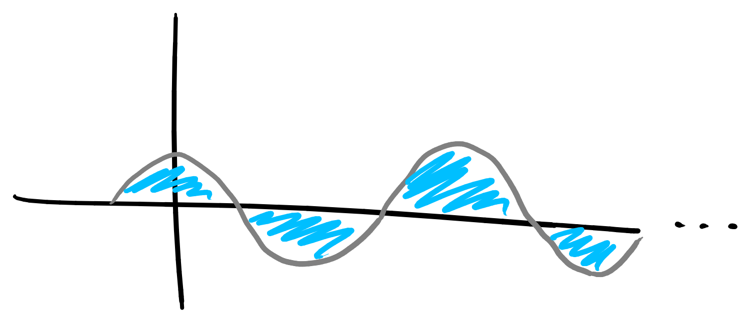
$\int$ is <u>divergent</u>!

A general rule:

$$\int_0^a \frac{1}{x^p}\,dx \quad \text{is:} \quad \begin{array}{l} \underline{\text{convergent}} \text{ if } p<1 \\ \underline{\text{divergent}} \text{ if } p \geq 1 \end{array}$$



Ex $\displaystyle\int_0^\infty \cos x\,dx$



$$= \lim_{t\to\infty} \int_0^t \cos x\,dx$$

$$= \lim_{t\to\infty} \sin x \Big|_0^t = \lim_{t\to\infty} \sin t \quad \underline{\text{does not exist}}$$

i.e. $\displaystyle\int_0^\infty \cos x\,dx$ diverges