Lecture 30

Apr 5 2010

Exam 7–9p tomorrow WEL 1.316

Topics:

- $\int$ by parts
- Trigonometric integrals
- Trigonometric substitutions
- Partial fractions
- Indeterminate forms and L'H rule
- Improper integrals
- Partial derivatives
- Iterated integrals
- Double integrals
- Sequences

Trig substitution

general pattern:

$\sqrt{a^2-x^2}$: try $x = a\sin\theta$

$\sqrt{x^2-a^2}$: try $x = a\sec\theta$

$\sqrt{x^2+a^2}$: try $x = a\tan\theta$

Ex $$\int_{3/2}^{3} \frac{\sqrt{9-x^2}}{x^2}\,dx$$

try $x = 3\sin\theta$ $dx = 3\cos\theta\, d\theta$

$x = \frac{3}{2} = 3\sin\theta$
$\to \sin\theta = \frac{1}{2}$
$\to \theta = \frac{\pi}{6}$

$x = 3 \to \sin\theta = 1$
$\to \theta = \frac{\pi}{2}$

then $$= \int_{\pi/6}^{\pi/2} \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cos\theta\, d\theta$$

$\sin^2\theta + \cos^2\theta = 1$

$$= \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cos\theta\, d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int_{\pi/6}^{\pi/2} \cot^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta \Big|_{\pi/6}^{\pi/2} = \underline{\underline{\sqrt{3} - \frac{\pi}{3}}}$$

Partial fractions

Start with $\int \frac{\text{polynomial}}{\text{polynomial}}\, dx$ and reduce it to, say,

$$\int \frac{3x+4}{x^2+1} + \frac{7}{x-4} + \frac{3}{x+2}\, dx$$

break up into 2 pieces:

$$\int \frac{3x}{x^2+1}\, dx + \int \frac{4}{x^2+1}\, dx$$

u-sub $u = x^2+1$

antiderv. is $4\tan^{-1} x$

do this by u-sub: $u = x-4$

$$\int \frac{7\,du}{u} = 7\ln(u)$$
$$= 7\ln(x-4)$$

$$\int \frac{8}{x^2+9}\, dx$$

$x^2+9 \rightsquigarrow x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{3}\right)$

$$= \int \frac{8}{9(\tan^2\theta+1)} 3\sec^2\theta\, d\theta = \frac{8}{9}\int \frac{3\sec^2\theta}{\sec^2\theta}\, d\theta = \frac{8}{3}\int d\theta = \frac{8}{3}(\theta) = \frac{8}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

[or just u-sub: $u = \frac{x}{3} \longrightarrow \int \frac{8\cdot 3\, du}{9(u^2+1)} = \frac{8}{3}\int \frac{du}{u^2+1} = \frac{8}{3}\tan^{-1}(u) = \frac{8}{3}\tan^{-1}\left(\frac{x}{3}\right)$]

check: $\frac{d}{dx}\left(\frac{8}{3}\tan^{-1}\left(\frac{x}{3}\right)\right) = \frac{8}{3}\,\frac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\frac{1}{3}$

$$= \frac{8}{9}\cdot\frac{1}{1+\frac{1}{9}x^2} = \frac{8}{9+x^2} \quad \checkmark$$

Indeterminate forms

When we try to take a limit and get s.t. like

$0\cdot\infty,\ \frac{0}{0},\ \frac{\infty}{\infty},\ \infty-\infty,\ 1^\infty \ldots$

call it indeterminate form, need some trick:

e.g. $\lim_{x\to\infty} \frac{x^2-4}{3x^2+x-7}$ $\frac{\infty}{\infty}$ divide top, bottom by x^2

$$= \lim_{x\to\infty} \frac{1-\frac{4}{x^2}}{3+\frac{1}{x}-\frac{7}{x^2}} = \frac{1}{3}$$

e.g. $\lim_{t\to 0} \frac{3e^t-3}{t} = \lim_{t\to 0} \frac{3e^t}{1} = \frac{3}{1} = \underline{\underline{3}}$ (L'H rule)

$\frac{0}{0}$

e.g. for 1^∞: like $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$. Take log:

Write $L = \ln \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x$

$$= \lim_{x\to\infty} x \ln\left(1+\frac{2}{x}\right) \to \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\left(\frac{1}{x}\right)} \to \frac{0}{0}$$

then use L'H: $= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\left(-\frac{2}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \underline{\underline{2}}$

So the answer to the original limit is $e^L = \underline{\underline{e^2}}$.

<u>Long division</u>: In partial frac. when num. is not lower degree than denominator

e.g. $$\int \frac{x^3+4x}{x^2-1}\,dx$$

$$\begin{array}{r|l} & x \\ \hline x^2+0x-1 & x^3+0x^2+4x \\ & x^3+0x^2-x \\ \hline & \quad 5x \end{array}$$

$$\frac{x^3+4x}{x^2-1} = x + \frac{5x}{x^2-1}$$

<u>Sequences</u>

$a_n = \frac{1}{n^2} + \frac{1}{n} + n^2 e^{-n}$: think about

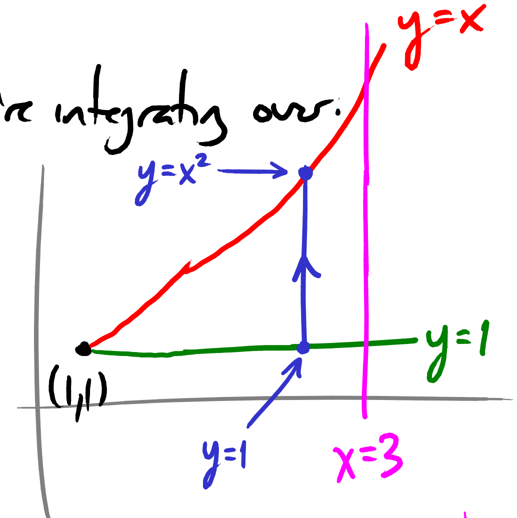
$$f(x) = \frac{1}{x^2} + \frac{1}{x} + x^2 e^{-x}$$

$\lim_{x\to\infty} f(x) = 0$ [using L'H rule!] <u>so</u> $\lim_{n\to\infty} a_n = 0$

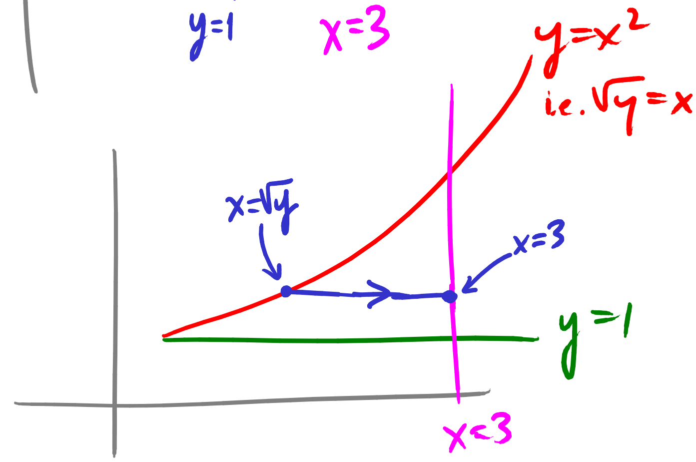
Reversing order of integrals:

$$\int_1^3 \left[\int_1^{x^2} F(x,y)\,dy\right] dx$$

Draw a picture of the region you're integrating over:



Then draw the horizontal slicing:

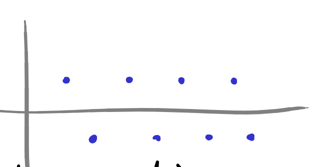


$$\int_1^9 \left[\int_{\sqrt{y}}^{3} F(x,y)\,dx\right]$$

A couple more sequences:

$$a_n = (-1)^n \frac{n^2+1}{3n^2+2} = (-1)^n \cdot \frac{1+\frac{1}{n^2} \nearrow 0}{3+\frac{2}{n^2} \searrow 0} \sim (-1)^n \cdot \frac{1}{3}$$ diverges

(oscillates between $+\frac{1}{3}$ and $-\frac{1}{3}$)



$$a_n = \frac{3n^2+(-1)^n n}{7n^2+\cos n} = \frac{3+\frac{1}{n}\cdot(-1)^n \nearrow 0}{7+\frac{1}{n^2}\cos(n) \searrow 0}$$ converges to $\frac{3}{7}$