Lecture 36

19 Apr 2010

Ex (from Lecture 34):

For what p does the series

$$\sum_{n=7}^{\infty} \left(\frac{n+1}{n^{5p}}\right) \cos(2\pi n)$$

converge?

First observation: $\cos(2\pi n) = 1$. So the series is <u>really</u>

$$\sum_{n=7}^{\infty} \frac{n+1}{n^{5p}}$$

Now at large n this would go $\sim \frac{n}{n^{5p}} = \frac{1}{n^{5p-1}}$

So try the Limit Comparison Test: using

$$a_n = \frac{n+1}{n^{5p}}, \quad b_n = \frac{1}{n^{5p-1}}$$

To see if the test applies:

calculate $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\left(\frac{n+1}{n^{5p}}\right)}{\left(\frac{1}{n^{5p-1}}\right)} = \lim\limits_{n\to\infty} 1 + \frac{1}{n} = 1$

So the test applies: $\sum a_n$ converges if and only if $\sum b_n$ converges.

$\sum b_n = \sum \frac{1}{n^{5p-1}}$: use <u>p-test</u> — <u>converges</u> if $5p-1>1$

i.e $p > \frac{2}{5}$

So finally, $\sum a_n$ converges if $p > \frac{2}{5}$

diverges if $p \leq \frac{2}{5}$

Absolute Convergence

$\sum a_n$

Call $\sum a_n$ "absolutely convergent" if $\sum |a_n|$ is convergent.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots$ $\quad \left[a_n = \frac{(-1)^n}{n^2} \right]$

$\displaystyle\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by p-test, $p = 2 > 1$)

So $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Fact: If $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.

If $\sum a_n$ is convergent but $\sum a_n$ is not absolutely convergent, then we call $\sum a_n$ conditionally convergent.

Ex $\sum (-1)^n \cdot \frac{1}{n}$ is convergent (by alt. series test)

But $\sum (-1)^n \cdot \frac{1}{n}$ is not absolutely convergent (because $\sum_1^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_1^{\infty} \frac{1}{n}$ is divergent (by p-test))

So $\sum_1^{\infty} (-1)^n \cdot \frac{1}{n}$ is <u>conditionally convergent</u>

So have 3 possibilities:

- absolutely convergent
- conditionally convergent
- divergent

<u>Ex</u> $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Has both <u>positive</u> and <u>negative</u> terms:

$+, -, -, -, +, +, +, -, \ldots$

<u>Not</u> alternating.

<u>Is</u> it <u>absolutely</u> convergent? Look at $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n^2}\right|$

We know $\left|\frac{\cos(n)}{n^2}\right| = \frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}$

And we know $\sum_1^{\infty} \frac{1}{n^2}$ converges (p-test)

So $\sum \left|\frac{\cos(n)}{n^2}\right|$ converges by Comparison Test $\left[\begin{array}{l} a_n = \left|\frac{\cos(n)}{n^2}\right| \\ b_n = \frac{1}{n^2} \end{array}\right]$

So $\sum_1^{\infty} \frac{\cos(n)}{n^2}$ <u>converges absolutely</u>

(So $\sum \frac{\cos(n)}{n^2}$ converges).

<u>Ex</u> $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ This is <u>alternating series</u> with $b_n = \frac{1}{\ln n}$.

So by alternating series test, it <u>converges</u>.

Does it converge <u>absolutely</u>?

i.e. does $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ <u>converge</u>?

$\frac{1}{\ln n} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges —

so $\sum \frac{1}{\ln n}$ diverges by Comparison Test.

So $\sum (-1)^n \frac{1}{\ln n}$ <u>converges conditionally</u>.

Ratio Test

1) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$

then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ (or $= \infty$)

then $\sum_{n=1}^{\infty} a_n$ is divergent.

[If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ then the test is inconclusive.]

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$. Ratio test: $a_n = (-1)^n \frac{n^3}{3^n}$

$a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^{n+1}} \cdot \frac{(n+1)^3}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 = \frac{1}{3}$$

Since $L = \frac{1}{3} < 1$, $\sum a_n$ converges absolutely by Ratio Test.