Lecture 38 | 23 Apr 2010

Housekeeping:

Exam 3 Tue May 4 7–9pm WEL 2.246

covers Ch 12.1–12.11

NO CALCULATORS!

Strategy tip:

If the question is "Does this series $\sum a_n$
1) converge absolutely
2) converge conditionally
3) diverge "

you should check first whether $\sum |a_n|$ converges!
If it does, the answer is 1).

Power Series (Ch 12.8)

A power series centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

(a will be a constant, x is a variable)

Ex $$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

is a power series centered at $a=0$.

Fact: For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are 3 possibilities:

Case 1) Series converges only when $x=a$.

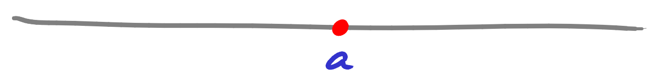
Case 2) Series converges for all values of x.

Case 3) There is some number $R>0$ such that
the series <u>converges</u> for $|x-a|<R$
the series <u>diverges</u> for $|x-a|>R$

We call R the "radius of convergence" of the series.
In case 1, we say $R=0$.
In case 2, we say $R=\infty$.

The "interval of convergence" is the set of all x where the series converges.

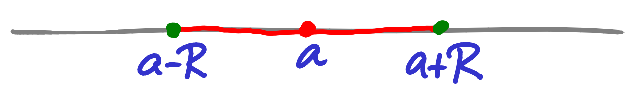
In case 1, it is just the point $x=a$.



In case 2, it is the whole line $-\infty<x<\infty$

or $x \in (-\infty, \infty)$.

In case 3, there are 4 possibilities for the interval:



$(a-R, a+R)$
$[a-R, a+R)$
$(a-R, a+R]$
$[a-R, a+R]$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = (x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} + \cdots$

Power series centered at $a=3$.

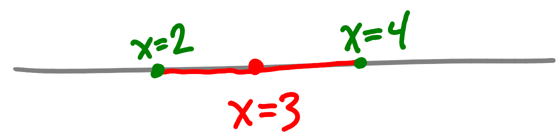
What is the interval of convergence?

Use Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|^{n+1}/_{n+1}}{|x-3|^n/_{n}} = |x-3| \cdot \frac{n}{n+1} \longrightarrow |x-3| \quad \text{as } n \to \infty$$

So series converges if $|x-3| < 1$

diverges if $|x-3| > 1$



So the radius of convergence $R = 1$

Does the series converge at the endpoints $x=2$ or $x=4$?

Ratio test is inconclusive there $(L=1)$

$x=4$: series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ divergent (p-test)

$x=2$: series is $\displaystyle\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergent (alt. series test)

Now we're done: interval of convergence is $[2,4)$

i.e. $2 \leq x < 4$

Ex $\displaystyle\sum_{n=1}^{\infty} n!\,x^n$ Power series centered at $a=0$.

Ratio Test: $$\frac{|a_{n+1}|}{|a_n|} = \frac{|(n+1)!\,x^{n+1}|}{|n!\,x^n|} = \frac{|(n+1)\,n!\,x^{n+1}|}{|n!\,x^n|}$$

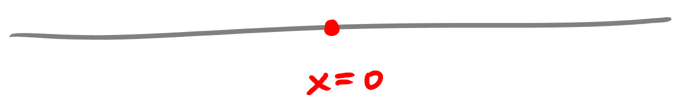
$$= (n+1)\frac{n!}{n!}\frac{|x|^{n+1}}{|x|^n} = (n+1)|x|$$

As $n\to\infty$, $$\lim_{n\to\infty}(n+1)|x| = \begin{cases} \lim_{n\to\infty} 0 & \text{if } x=0 \\ \lim_{n\to\infty}(n+1)|x| & \text{if } x\neq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x=0 \\ \infty & \text{for all } x\neq 0 \end{cases}$$

So the power series converges for $x=0$, diverges for all $x\neq 0$.

Radius of convergence $R=0$.



Interval of convergence is just a single point, $x=0$. (Case 1)

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ ("Bessel function")

Ratio test: $$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+2}}{2^{2n+2}((n+1)!)^2}\cdot\frac{2^{2n}(n!)^2}{|x|^{2n}}$$

$$= \frac{|x|^2}{4}\cdot\left[\frac{n!}{(n+1)!}\right]^2 = \frac{|x|^2}{4}\cdot\left(\frac{1}{n+1}\right)^2 \to 0 \text{ as } n\to\infty$$

$L = 0 < 1$, so series converges — <u>for all</u> x!

Radius of convergence is $R = \infty$

Interval of convergence is $(-\infty, \infty)$.

<u>Ex</u> $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ Power series centered at $a=-2$.

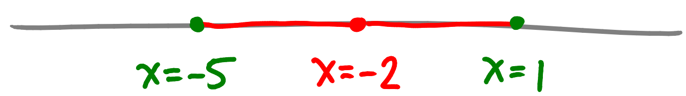
Ratio test: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\,|x+2|^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\,|x+2|^n}$

$\displaystyle = \left(\frac{n+1}{3n}\right)|x+2| \longrightarrow \frac{1}{3}|x+2| \quad \text{as } n\to\infty$

So series <u>converges</u> if $\frac{1}{3}|x+2| < 1$, i.e. $|x+2| < 3$

<u>diverges</u> if $|x+2| > 3$

Radius of convergence $R = 3$



Does it converge at the ends $x=-5$ and $x=1$?

$x=1$: $\displaystyle\sum \frac{n\,3^n}{3^{n+1}} = \sum \frac{n}{3}$ diverges (Test For Div.)

$x=-5$: $\displaystyle\sum \frac{n(-3)^n}{3^{n+1}} = \sum \frac{n}{3}(-1)^n$ diverges (Test For Div.)

So the interval of convergence is $\underline{\underline{(-5, 1)}}$