Lecture 4

Admin: HW03 available now, due next Tue 3am

LM usually due M, W, Sat nights midnight

but, <u>no LM</u> <u>this W</u>!

Tricky substitutions:

1) $\int \frac{dx}{1+9x^2} = ?$ Idea: we know $\int \frac{du}{1+u^2} = \tan^{-1} u + C$

So, take $\underline{u = 3x}$ (because then $u^2 = 9x^2$)

then $du = 3\,dx$

$\frac{1}{3} du = dx$

$$\int \frac{dx}{1+9x^2} = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1}(3x) + C$$

(could similarly do e.g. $\int \frac{dx}{\sqrt{1-16x^2}}$ by $u = 4x$...)

2) $\int \frac{5}{x^2+6x+10}\,dx$ try to relate this to $\int \frac{1}{u^2+1}\,du$

the trick: "completing the square"

take $u = x+3$ then $u^2 = (x+3)^2 = x^2 + 6x + 9$

$du = dx$

so

$$\int \frac{5}{x^2+6x+10}\,dx = \int \frac{5}{(x+3)^2+1}\,dx$$

$$= \int \frac{5}{u^2+1}\,du$$

$$= 5\tan^{-1}(u) + C \quad \cdots$$

Q $\displaystyle\int \frac{1}{x\ln x}\,dx = ?$ $\qquad \ln|\ln x| + C$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$$= \int \frac{du}{u} = \ln|u| + C$$
$$= \underline{\ln|\ln x| + C}$$

why do we say
$\displaystyle\int \frac{1}{t}\,dt = \ln|t|$
instead of $\ln t$?

$$\lim_{x\to\infty} \ln|\ln x| = \infty$$

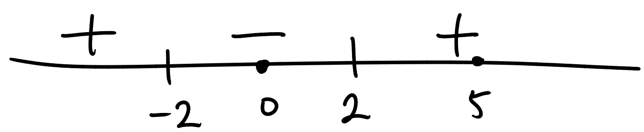
$f(x) = \ln|\ln x|$

$f(10^{100}) \approx 5.4\ldots$

Q $\displaystyle\int_0^5 |x^2-4|\,dx$

to do this kind of integral:

<u>split it up</u> into domains where $x^2-4>0$
or $x^2-4<0$



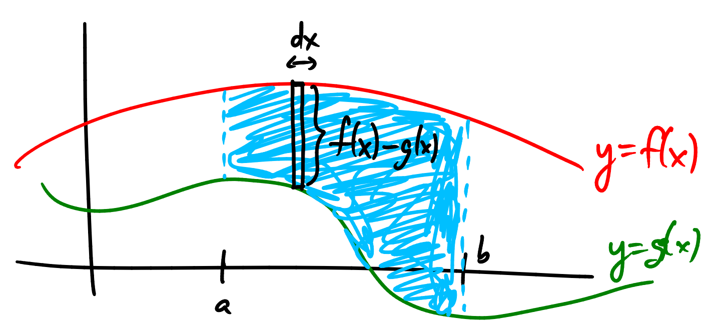
then, $\displaystyle\int_0^5 |x^2-4|\,dx = \int_0^2 |x^2-4|\,dx + \int_2^5 |x^2-4|\,dx$

$$= \int_0^2 4-x^2\,dx + \int_2^5 x^2-4\,dx$$

$$= \ldots$$

<u>Areas between curves</u>

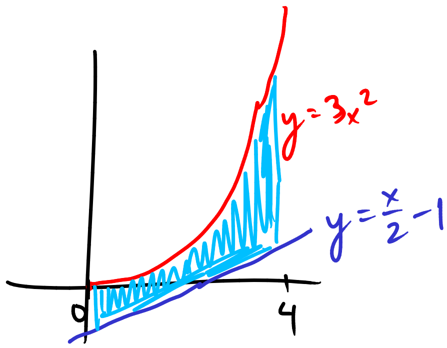
Two curves $y=f(x)$ and $y=g(x)$.
Say $f(x) > g(x)$ for x in $[a,b]$.



Then the area of the shaded region is $(f(x)-g(x))\,dx$

So the total area is
$$A=\int_a^b (f(x)-g(x))\,dx$$

Q Find area between $y=3x^2$ and $y=\frac{x}{2}-1$ with x between 0 and 4.



$$\int_0^4 \underbrace{(3x^2)}_{\text{top}} - \underbrace{\left(\frac{x}{2}-1\right)}_{\text{bottom}}\,dx$$

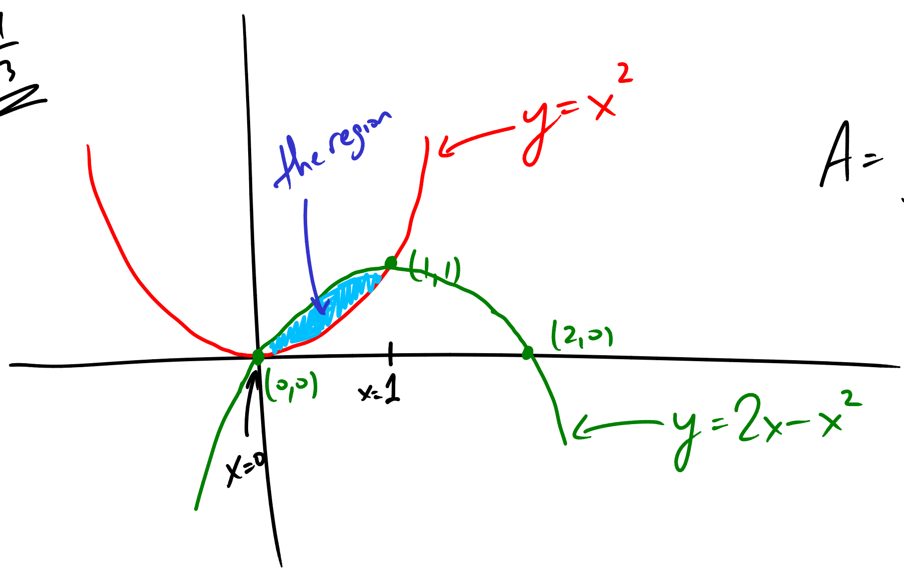
$$= x^3-\frac{x^2}{4}+x\,\Big|_0^4$$

$$=(64-4+4)-(0+0+0)$$

$$=\underline{64}$$

Q Find the area of the region between $y=x^2$ and $y=2x-x^2$.

$\underline{\frac{1}{3}}$



$$A=\int_0^1 \Big(\underbrace{(2x-x^2)}_{\text{top}}-\underbrace{x^2}_{\text{bottom}}\Big)\,dx$$

$$=\int_0^1 2x-2x^2\,dx$$

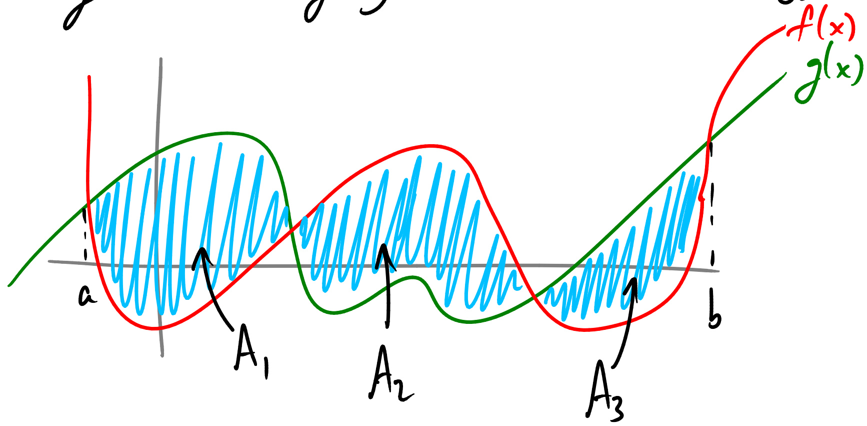
$$=x^2-\frac{2}{3}x^3\,\Big|_0^1$$

$$=\left(1-\frac{2}{3}\right)-(0-0)$$

$$=\underline{\frac{1}{3}}$$

A rule that finds the area between $y=f(x)$ and $y=g(x)$ no matter which is bigger:

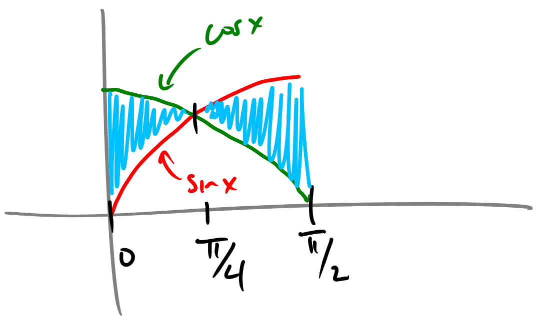
$$A = \int_a^b |f(x)-g(x)|\, dx$$



$$A = A_1 + A_2 + A_3$$

Q Find the area of the region between $y=\sin x$ and $y=\cos x$ when x ranges between $x=0$ and $x=\pi/2$.

$2 \quad 2\sqrt{2}-2 \quad 0$



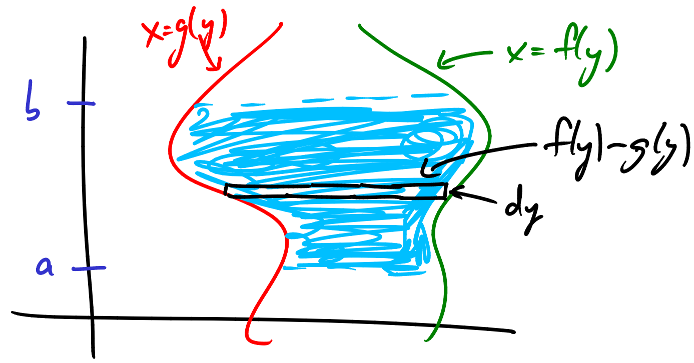
$$\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$\int_0^{\pi/4} (\cos x - \sin x)\, dx = \sin x + \cos x \Big|_0^{\pi/4} = \left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right)$$
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1)$$
$$= \sqrt{2}-1$$

the other part $\int_{\pi/4}^{\pi/2} \sin x - \cos x\, dx = \sqrt{2}-1$ also

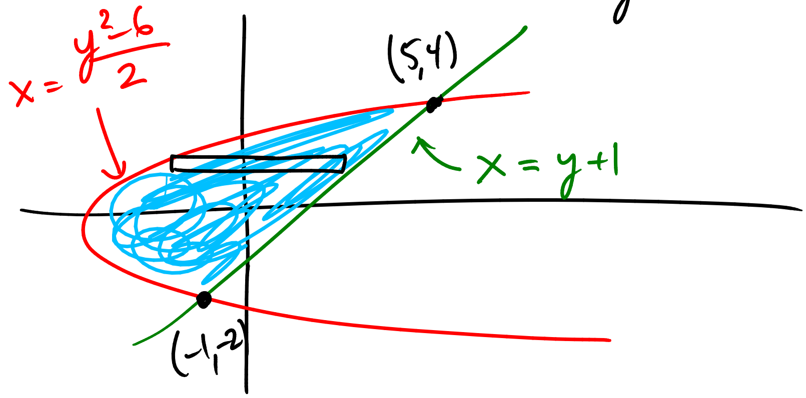
so total $= \underline{\underline{2\sqrt{2}-2}}$

Can also consider curves: $x=f(y)$, $x=g(y)$



$$A = \int_a^b (f(y)-g(y))\, dy$$

Q. Find the area between the parabola $y^2 = 2x + 6$ and line $y = x - 1$.



$$A = \int_{-2}^{4} (y+1) - \left(\frac{y^2-6}{2}\right) dy$$

$$= \dots = \underline{18}$$