Lecture 6

Integration by parts

Product rule: $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$

Take $\int dx$ of both sides:

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\Longrightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Or: write $u = f(x)$, $du = f'(x)\,dx$, $v = g(x)$, $dv = g'(x)\,dx$

then $\boxed{\int u\,dv = uv - \int v\,du}$

$\xleftarrow{u}$ $\xrightarrow{dv}$

LIATE

L: logs, I: inv trig, A: algebraic, T: trig, E: exp

Q Find $\int \underset{u}{x}\,\underbrace{\cos(5x)\,dx}_{dv}$.

$u = x$ $\quad v = \frac{1}{5}\sin(5x)$

$du = dx$ $\quad dv = \cos(5x)\,dx$

$$= uv - \int v\,du$$

$$= x \cdot \frac{1}{5}\sin(5x) - \int \frac{1}{5}\sin(5x)\cdot dx$$

$$= \underline{\frac{x}{5}\sin(5x) + \frac{1}{25}\cos(5x) + C}$$

Q Find $\int \ln x\,dx$.

$\ln x\,dx = u\,dv$

$u = \ln x$ $\quad v = x$

$du = \frac{1}{x}\,dx$ $\quad dv = dx$

$\left(\frac{du}{dx} = \frac{1}{x}\right)$

$$\int \ln x\,dx = uv - \int v\,du$$

$$= x\ln x - \int x \cdot \frac{1}{x}\,dx$$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x + C$

LIATE

t^2 e^t

Q Find $\int e^t t^2 \, dt$.

$u = t^2$ $v = e^t$

$du = 2t\,dt$ $dv = e^t dt$

$= uv - \int v \, du$

$= t^2 e^t - \int e^t 2t \, dt$

DO IBP AGAIN:

$= t^2 e^t - 2\int e^t t \, dt$

$u = t$ $v = e^t$

$du = dt$ $dv = e^t dt$

$= t^2 e^t - 2\left(uv - \int v \, du\right)$

$= t^2 e^t - 2\left(te^t - \int e^t dt\right)$

$= t^2 e^t - 2te^t + 2e^t + C$

Ex $\int e^x \sin x \, dx$

$u = \sin x$ $v = e^x$

$du = \cos x \, dx$ $dv = e^x dx$

$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

$u = \cos x$ $v = e^x$

$du = -\sin x \, dx$ $dv = e^x dx$

$= e^x \sin x - \left(e^x \cos x - \int -e^x \sin x \, dx\right)$

IBP twice

$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

$\longrightarrow 2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$

$\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right)$

Q $\int_0^{\pi} t\sin(3t)\,dt$

$u = t \qquad v = -\frac{1}{3}\cos(3t)$

$du = dt \qquad dv = \sin(3t)\,dt$

$$\int_0^{\pi} t\sin(3t)\,dt = uv\Big|_{t=0}^{t=\pi} - \int_{t=0}^{t=\pi} v\,du$$

$$= (t)\left(-\frac{1}{3}\cos 3t\right)\Big|_0^{\pi} - \int_0^{\pi} -\frac{1}{3}\cos(3t)\,dt$$

$$= \pi\cdot\left(-\frac{1}{3}\cos 3\pi\right) - 0\cdot\left(-\frac{1}{3}\cos 0\right) + \frac{\sin(3t)}{9}\Big|_0^{\pi}$$

$$= \pi\cdot\left(-\frac{1}{3}\right)(-1) - 0 + \frac{1}{9}(\sin 3\pi - \sin 0)$$

$$= \frac{\pi}{3}$$

Q $\int \cos(\sqrt{x})\,dx = ?$

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\,dx$

$2\sqrt{x}\,dt = dx$

$$= \int \cos(t)\,dx$$

$$= \int \cos(t)\cdot 2\sqrt{x}\,dt$$

$$= \int \underbrace{\cos(t)}_{dv}\cdot \underbrace{2t}_{u}\,dt$$

IBP: $u = 2t \qquad v = \sin t$

$du = 2dt \qquad dv = \cos t\,dt$

$$= uv - \int v\,du = 2t\sin t - \int \sin t\,(2\,dt)$$

$$= 2t\sin t + 2\cos t + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

① $\displaystyle\int_{\theta=\sqrt{\frac{\pi}{2}}}^{\theta=\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta$

try by parts: $u = \cos(\theta^2)$ $\quad v = \frac{1}{4}\theta^4$

$du = -2\theta \sin(\theta^2) d\theta$ $\quad dv = \theta^3\, d\theta$

$\rightarrow \int v\, du = \int \theta^3 \cdot (-2\theta \sin\theta^2)\, d\theta$

$= -2\int \theta^4 \sin\theta^2\, d\theta$

$\longrightarrow$ NO HELP

try subst: $t = \theta^2$ $\quad dt = 2\theta\, d\theta$ $\quad d\theta = \frac{dt}{2\theta}$

$$\int_{t=\pi/2}^{t=\pi} \theta^3 \cos(t) \frac{dt}{2\theta}$$

$$= \int_{t=\pi/2}^{t=\pi} \frac{1}{2} \theta^2 \cos(t)\, dt$$

$$= \int_{\pi/2}^{\pi} \frac{1}{2} t \cos(t)\, dt$$

OR, IBP $\displaystyle\int \theta^3 \cos(\theta^2)\, d\theta$ $\quad u = \theta^2$ $\quad v = \frac{1}{2}\sin(\theta^2) d\theta$

$du = 2\theta\, d\theta$ $\quad dv = \theta \cos(\theta^2)\, d\theta$

$\cdots \quad -\frac{1}{2} - \frac{\pi}{4}$