Lecture 7

Next LM already posted, due Mon night midnight

Exam 1 Tue Oct 3 7–9pm covers Sec 5.3–7.1
Jester A121A (from start to int by parts)

Trigonometric integrals

$\int \sin^5\theta \cos\theta \, d\theta = ?$ substitution: $u = \sin\theta$, $du = \cos\theta \, d\theta$

$= \int u^5 \, du$

$= \frac{1}{6}u^6 + C = \frac{1}{6}(\sin\theta)^6 + C = \frac{1}{6}\sin^6\theta + C$

Similarly for $\int \sin^a\theta \cos\theta \, d\theta$ any a

or for $\int \cos^b\theta \sin\theta \, d\theta$ (use $u = \cos\theta$)

What about: Q $\int \sin^3\theta \, d\theta = ? = \int \sin\theta \sin^2\theta \, d\theta$

$\int \sin(1-\cos^2 x)$

$u = \cos x$
$du = -\sin x$

$\int \cancel{\sin}(1-u^2)\frac{du}{\cancel{-\sin x}}$

$-1\int 1-u^2 \, du$

$-1\left(u - \frac{u^3}{3}\right) + C$

$f(x) = -1\cos(x) + \frac{\cos^3(x)}{3} + C$

Q: could we also use a trig id?

$$\int \sin^3\theta \, d\theta = \int \sin\theta \cdot \sin^2\theta \, d\theta$$

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$= \int \sin\theta \cdot \tfrac{1}{2}(1-\cos 2\theta)\, d\theta$$

doesn't help

Q $\int \sin^5\theta \cos^2\theta \, d\theta = ?$

either reduce to $\int \sin^k\theta \cos\theta \, d\theta$ or $\int \cos^k\theta \sin\theta \, d\theta$

$$= \int \sin^4\theta \cos^2\theta \, (\sin\theta \, d\theta)$$

$$= \int (\sin^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$= \int (1-\cos^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$$

$$= \int (1-u^2)^2 \, u^2 \, (-du)$$

$$= \int (1-2u^2+u^4)\, u^2 \, (-du)$$

$$= -\int u^2 - 2u^4 + u^6 \, du$$

$$= -\frac{u^3}{3} + \frac{2}{5}u^5 - \frac{u^7}{7} = -\frac{\cos^3\theta}{3} + \frac{2}{5}\cos^5\theta - \frac{1}{7}\cos^7\theta + C$$

General rule for $\int \sin^a\theta \cos^b\theta \, d\theta$:

If a is odd, pick off one $\sin\theta$, write $(\sin\theta \, d\theta)$
use $\sin^2\theta = 1-\cos^2\theta$ to eliminate rest of $\sin\theta$'s, then use $u = \cos\theta$.

If b is odd, pick off one $\cos\theta$, write $(\cos\theta \, d\theta)$, use $\cos^2\theta = 1 - \sin^2\theta$ to elim. rest of $\cos\theta$'s, then $u = \sin\theta$.

What about even power?

Half-angle formulas:

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

Q $\int \sin^2\theta \, d\theta$

$\left(\text{and. } \int_0^\pi \sin^2\theta \, d\theta = ? \right)$

$$\int \sin^2\theta \, d\theta = \int \frac{1}{2}(1-\cos 2\theta) = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C$$

$$\int_0^\pi \sin^2\theta \, d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta \Big|_0^\pi = \frac{\pi}{2}$$

ie "the average value of $\sin^2$ from 0 to π is $\frac{1}{2}$"

$\sin^2 + \cos^2 = 1$

Q $\int \cos^4\theta \, d\theta = ?$

$= \int (\cos^2\theta)^2 \, d\theta$

$= \int \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$

$= \frac{1}{4}\int (1+\cos 2\theta)^2 \, d\theta$

$= \frac{1}{4}\int 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$

$= \frac{1}{4}\int 1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta) \, d\theta$

$= \dots = \frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C$

Ex $\int \tan^6 x \sec^4 x \, dx = ?$ $\qquad \left(\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array} \qquad \sec^2 x = 1 + \tan^2 x \right]$

$= \int \tan^6 x \sec^2 x \cdot \sec^2 x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u^6 (1 + u^2) \, du$

$= \frac{u^7}{7} + \frac{u^9}{9} + C \quad = \underline{\underline{\frac{\tan^7}{7} + \frac{\tan^9}{9} + C}}$

Same strategy for $\tan^a x \sec^b x$ when b is <u>even</u>

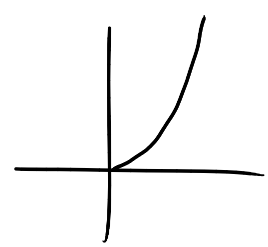
Ex $\int \tan^3 x \sec^5 x \, dx$ $\qquad \left[\begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array} \right]$

$= \int \tan^2 x \sec^4 x \, (\sec x \tan x \, dx)$

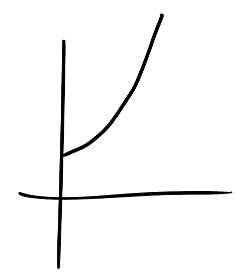
$= \int (u^2 - 1) \, u^4 \, du$

$= \dots = \frac{u^7}{7} - \frac{u^5}{5} = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5}$

tan x



sec x



$\frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2} \longrightarrow \tan^2 + 1 = \sec^2$

$\tan^2 = \sec^2 - 1$

Same strat. for
$\int \tan^a x \sec^b x$
and a is <u>odd</u>, $b \geq 1$

Handy facts: $\int \tan x \, dx = \ln|\sec x| + C$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Ex $\displaystyle\int \tan^3 x \, dx = \int \tan x (\sec^2 x - 1) \, dx$

$$= \int \tan x \sec^2 x \, dx \quad - \int \tan x \, dx$$

$$\left[\begin{array}{l} u = \tan x \\ du = \sec^2 x \\ \quad \int u \, du \\ \quad = \frac{1}{2}u^2 \\ \quad = \frac{1}{2}\tan^2 x \end{array}\right]$$

$$= \underline{\underline{\frac{1}{2}\tan^2 x \quad - \ln|\sec x| + C}}$$

Ex $\displaystyle\int \sin 4x \cos 7x \, dx$

Use product-to-sum identities:

$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

so here

$$= \frac{1}{2}\int \sin(-3x) + \sin(11x) \, dx$$

$$= \underline{\underline{\frac{1}{2}\left(\frac{1}{3}\cos 3x - \frac{1}{11}\cos 11x\right) + C}}$$