Lecture 8

Exam 1 next Tue (1 week from today) 7–9pm Jester A121A

no calculators

all you should bring: pencils, ID

review Thu in class

HW05 due Tue 3am

exam covers 5.3–7.1
(beginning of course to int. by parts)
(HW 1,2,3, part of 4)

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re trig identities: for Exam 2 you should know:

$\sin^2\theta + \cos^2\theta = 1$, double-angle identities (/half-angle)

Last time: trigonometric integrals

like $\int \sin^a\theta \cos^b\theta \, d\theta$

$\int \sec^a\theta \tan^b\theta \, d\theta$

$d(\sin\theta) = \cos\theta \, d\theta$
$d(\cos\theta) = -\sin\theta \, d\theta$

$d(\sec\theta) = \sec\theta \tan\theta \, d\theta$
$d(\tan\theta) = \sec^2\theta \, d\theta$

One more (tricky) example:

$$\int \sec^3 x \, dx$$

might try: $\int \sec x \underbrace{(\sec^2 x \, dx)}_{du}$ $\quad u = \tan x$

but this looks hard — hard to write $\sec x$ in terms of u

Instead, use IBP: $u = \sec x$ $\quad v = \tan x$
$du = \sec x \tan x \, dx$ $\quad dv = \sec^2 x \, dx$

then $\int \sec^3 x\,dx = \sec x \tan x - \int \tan x (\sec x \tan x)\,dx$

$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$

use $\tan^2 x = \sec^2 x - 1$

so, this

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x (\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\underline{\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C}$$

Next method:

Trigonometric Substitution

Q $\int \frac{\sqrt{9-x^2}}{x^2}\,dx = ?$

Idea: want to get rid of the square root!

Want $9 - x^2 = (\text{something})^2$

remember $\cos^2 + \sin^2 = 1$

$1 - \sin^2 = \cos^2$

$1 - \sin^2\theta = \cos^2\theta$

$9 - 9\sin^2\theta = 9\cos^2\theta$

So, set $9\sin^2\theta = x^2$ ie $x = 3\sin\theta$

take $x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$\int \frac{\sqrt{9 - 9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\,d\theta$$

$$= \int \frac{\sqrt{9\cos^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta \, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta} \, d\theta = \int \cot^2\theta \, d\theta = \int (\csc^2\theta - 1) \, d\theta$$

$$= -\cot\theta - \theta + C$$

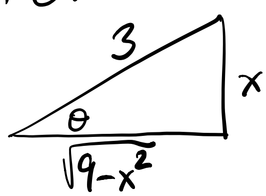
Now, want to rewrite this in terms of <u>original</u> variable x.

$x = 3\sin\theta$. what is $\cot\theta$?

$\sin\theta = \frac{x}{3}$

$\parallel$

$\frac{\text{opp}}{\text{hyp}}$



$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{9-x^2}}{x}$

what is θ? $x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

$\sin^{-1}\left(\frac{x}{3}\right) = \theta$

So <u>finally</u>, $\displaystyle \int \frac{\sqrt{9-x^2}}{x^2} \, dx = \underline{\underline{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}}$

<u>Q</u> $\displaystyle \int \frac{1}{x^2\sqrt{x^2+4}} \, dx = ?$

$x = 2\tan\theta$

<u>because</u> $\tan^2 + 1 = \sec^2$

$4\tan^2\theta + 4 = 4\sec^2\theta$

$\Rightarrow x = 2\tan\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$$\int \frac{1}{4\tan^2\theta \cdot \sqrt{4\sec^2\theta}} \, 2\sec^2\theta \, d\theta$$

$$= \int \frac{1}{4\tan^2\theta \, 2\sec\theta} 2\sec^2\theta \, d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta} d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

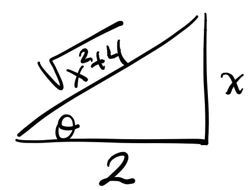
$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} d\theta \qquad \begin{aligned} u &= \sin\theta \\ du &= \cos\theta \, d\theta \end{aligned}$$

$$= \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) = -\frac{1}{4u} = -\frac{1}{4\sin\theta}$$

and $x = 2\tan\theta$

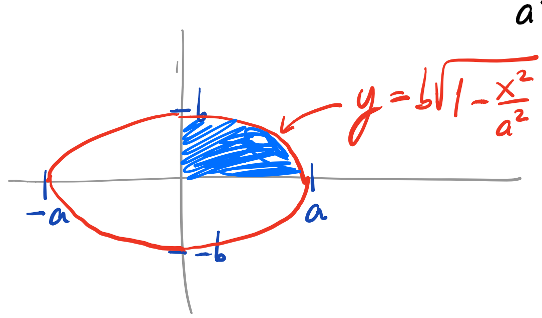
$\frac{x}{2} = \tan\theta$



$\longrightarrow \sin\theta = \frac{opp}{hyp} = \frac{x}{\sqrt{x^2+4}}$

So finally $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}} dx = -\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + C$

Q Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$\text{area} = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$1 - \sin^2 = \cos^2$

so we want $\sin^2\theta = \frac{x^2}{a^2}$

i.e. $\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

$$x = a\sin\theta$$
$$dx = a\cos\theta\, d\theta$$

$$\rightsquigarrow \quad 4b\int_0^a \sqrt{1-\frac{a^2\sin^2\theta}{a^2}}\; a\cos\theta\, d\theta$$

$$= 4ba\int_0^a \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$$

$$= 4ba\int_{x=0}^{x=a} \cos\theta\cdot\cos\theta\, d\theta$$

$$= 4ba\int_{\theta=0}^{\theta=\pi/2} \cos^2\theta\, d\theta$$

$x = a\sin\theta$

limits: $x=0 \to a\sin\theta = 0 \quad \theta = 0$

$x = a \to a\sin\theta = a$

$\sin\theta = 1 \quad \theta = \pi/2$

$$= 4ba\cdot\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= 4ba\cdot\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Big|_0^{\pi/2}$$

$$= 4ba\cdot\frac{\pi}{4} = \underline{\underline{\pi\cdot b\cdot a}}$$