Lecture 10

Midterm 1 tonight Jester A121A 7-9pm

bring pencils, ID come ~10-15min early

Q/ $\int \tan x \, dx = \ln|\sec x| + C$

$= -\ln|\cos x| + C$? which to use?

either: they're the same, because $|\sec x| = \frac{1}{|\cos x|} = |\cos x|^{-1}$

so $\ln|\sec x| = \ln\left(|\cos x|^{-1}\right) = -\ln|\cos x|$

Last time: trigonometric substitution — a way to simplify some integrals involving $\sqrt{}$

e.g. for $\int$ involving $\sqrt{1-x^2}$, substitute $x = \sin\theta$

so $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$ ← simpler

or for $\int \sqrt{16-x^2}$ $\quad x = 4\sin\theta$

$\int \sqrt{25+x^2}$ $\quad x = 5\tan\theta$ ...

$\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ put $x = \sin\theta$ $\quad dx = \cos\theta\, d\theta$

$$\int \frac{\sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}}\,dx = \int \frac{\sin^{-1}(\sin(\theta))}{\cos\theta}\cos\theta\, d\theta = \int \theta\, d\theta$$

Partial fractions

A method for integrating "complicated" rational functions $\frac{P(x)}{Q(x)}$

P, Q both polynomials

Ex $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

try to break this up into simpler rational f's (fractions)

Factor the denominator: $2x^3+3x^2-2x$

$$= x(2x^2+3x-2)$$
$$= x(2x-1)(x+2)$$

OK, we got distinct linear factors (no x^2, no $(--)^2$)

So: set $\dfrac{x^2+2x-1}{2x^3+3x^2-2x} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{x+2}$

To find A, B, C: mult. both sides by the denominator $x(2x-1)(x+2)$

$$x^2+2x-1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1)$$

now plug in $x=0$: then have $-1 = A(-1)(2)$ $\quad -1=-2A$, $A=\frac{1}{2}$

$x=\frac{1}{2}$: then have $\frac{1}{4}+1-1 = B(\frac{1}{2})(\frac{5}{2})$

$\frac{1}{4} = \frac{5B}{4}$ $\quad B=\frac{1}{5}$

$x=-2$: then have $4-4-1 = C(-2)(-5)$

$-1 = 10C \qquad C = -\frac{1}{10}$

So, $$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1/2}{x} + \frac{1/5}{2x-1} + \frac{-1/10}{x+2}$$

thus $$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx + -\frac{1}{10}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + K$$

$$\left(= \frac{1}{2}\ln|x| + \frac{1}{10}\ln\frac{|2x-1|}{|x+2|} + K \right)$$

What if the denominator doesn't factor completely into linear factors?

Ex $$\int \frac{2x^2-x+4}{x^3+4x}\,dx$$ Factor: $x^3+4x = x(x^2+4)$

Write $$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

To find A, B, C: mult. both sides by denom. $x^3+4x = x(x^2+4)$

$\hookrightarrow$ $$2x^2-x+4 = A(x^2+4) + (Bx+C)x$$

two methods. ① plug in values for x

$x=0 \rightarrow 4 = 4A$

$x=1 \rightarrow 5 = 5A+B+C$

$x=-1 \rightarrow 7 = 5A+B-C$

$\hookrightarrow A=1,\ B=1,\ C=-1$

② $2x^2-x+4 = Ax^2+4A+Bx^2+Cx$
$= (A+B)x^2+(C)x+4A$

$\longrightarrow$ $2 = A+B$
$-1 = C$ $\longrightarrow$ $A=1,\ C=-1,\ B=1$
$4 = 4A$

So, our $\int = \int \frac{A}{x} + \frac{Bx+C}{x^2+4}dx = \int \frac{1}{x} + \frac{(x-1)}{x^2+4}dx$

$= \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$

$\ln|x|$ — use $u=x^2+4$ get $\frac{1}{2}\ln|x^2+4|$ — use $x=2u$, $u=\frac{1}{2}x$ get $-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

$= \underline{\underline{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + K}}$

Q how to do $\int \frac{dx}{x^2+9}$? key fact: $\int \frac{dx}{1+x^2} = \tan^{-1}(x)$

So, substitute $x = 3u$
$dx = 3du$

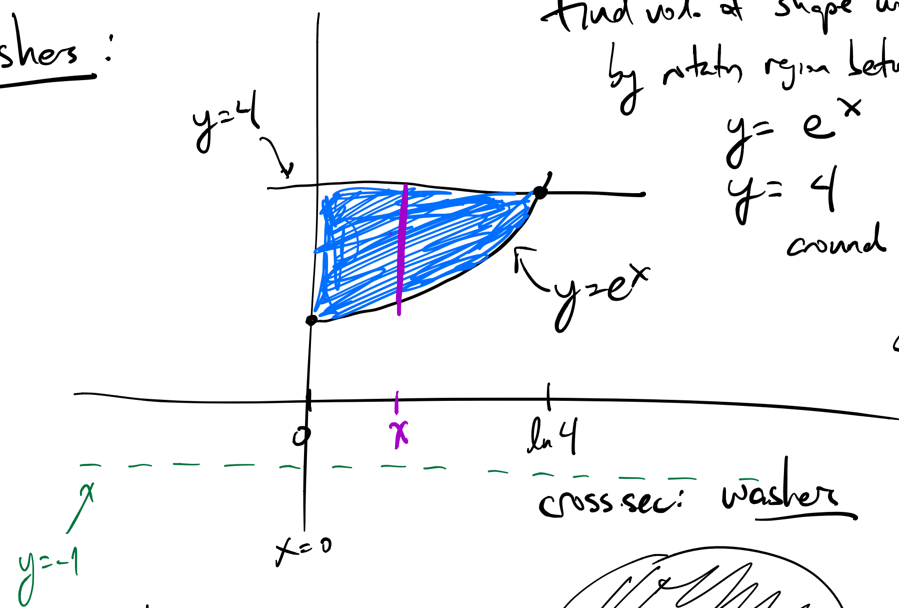
$\int \frac{3du}{(3u)^2+9} = \frac{3}{9}\int\frac{du}{u^2+1}$
$= \frac{1}{3}\tan^{-1}u$

$\int \frac{2x}{x^2-7}dx$ $u=x^2-7$

L'Hospital: $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$ $\underline{\underline{\text{if}}}$ $f(0)=0$, $g(0)=0$

<u>Washers</u>:

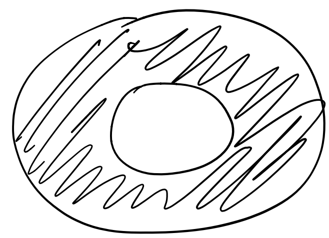
find vol. of shape we get by rotating region between $y = e^x$, $x=0$, $y=4$ around $\underline{x\text{-axis}}$.



$4 = e^x$

$\ln 4 = x$

cross sec: <u>washer</u>



inner radius $= e^x$

outer radius $= 4$

$$V = \pi \int_0^{\ln 4} 4^2 - (e^x)^2 \, dx$$

if around $y=-1$ instead:

$$V = \pi \int_0^{\ln 4} 5^2 - (e^x+1)^2 \, dx$$

$$\int_0^{1/2} \frac{3xe^{-x}}{(x-1)^2} \, dx$$

$u = 3xe^{-x}$ $\quad v = -\frac{1}{x-1}$

$du = 3e^{-x} - 3xe^{-x}$ $\quad dv = \frac{1}{(x-1)^2} \, dx$

$= 3(1-x)e^{-x}$

Rotating around y-axis:

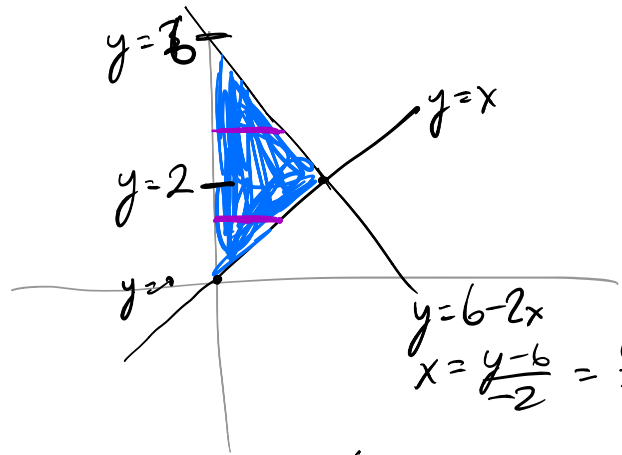
rotate shape bounded by

$y = x$

$y = 6-2x$

around y-axis.

cross sections: circles



$x = \frac{y-6}{-2} = \frac{6-y}{2}$

$$\int_0^2 \pi (y)^2\,dy + \int_2^6 \pi \left(\frac{6-y}{2}\right)^2 dy$$