Lecture 10

Midterm 1 today Jester A121A 7-9pm come 10-15min early
bring ID, pencils

Last time: trigonometric substitution — a method for simplifying integrals involving $\sqrt{}$.

e.g.

for $\int$ involving $\sqrt{1-x^2}$, set $x = \sin\theta$

then $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

Similarly for $\int$ inv. $\sqrt{1-36x^2}$, set $x = \frac{1}{6}\sin\theta$

then $\sqrt{1-36x^2} = \sqrt{1-36\left(\frac{1}{6}\sin\theta\right)^2}$

$= \sqrt{1-\sin^2\theta}$

$= \sqrt{\cos^2\theta} = \cos\theta$

how about $\displaystyle\int \frac{1}{(1-x^2)^{5/2}}\,dx$? set $x = \sin\theta$

$dx = \cos\theta\,d\theta$

then get $\displaystyle\int \frac{1}{(1-\sin^2\theta)^{5/2}}\cos\theta\,d\theta$

$$= \int \frac{1}{(\cos^2\theta)^{5/2}}\cos\theta\,d\theta$$

$$= \int \frac{1}{(\cos\theta)^5}\cos\theta\,d\theta$$

$$= \int \sec^4\theta\,d\theta$$

Partial fractions

How to integrate "complicated" rational functions $\frac{P(x)}{Q(x)}$ P, Q polynomials?

Ex $$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

Factor the denominator: $$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(2x-1)(x+2)$$

Linear factors, no repeated factor: this is the easiest case.

and $\underset{\overset{\shortparallel}{2}}{\text{degree(numerator)}} < \underset{\overset{\shortparallel}{3}}{\text{degree(denom)}}$

Then set $$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

Need to find A, B, C.

Multiply both sides by denom. $= x(2x-1)(x+2)$.

$$\longrightarrow \quad x^2+2x-1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1).$$

plug in values of x: $x=0 \rightarrow \quad -1 = A(2\cdot 0-1)(0+2) + 0 + 0$

i.e. $-1 = -2A$

$\frac{1}{2} = A$

$x=-2 \rightarrow \quad -1 = 0 + 0 + C(-2)(-5)$

$-1 = 10C$

$$-\tfrac{1}{10} = C$$

$$x = \tfrac{1}{2} \longrightarrow \tfrac{1}{4} = 0 + B\left(\tfrac{1}{2}\right)\left(\tfrac{5}{2}\right)$$

$$\tfrac{1}{4} = B \cdot \tfrac{5}{4}$$

$$\tfrac{1}{5} = B$$

So, our $\int$ is $\displaystyle\int \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}\,dx$

$$= \int \frac{1}{2}\cdot\frac{1}{x}\,dx + \int \frac{1}{5}\cdot\frac{1}{2x-1}\,dx + \int -\frac{1}{10}\cdot\frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + K$$

$$\frac{d}{dx}\ln|2x-1| = \frac{2}{2x-1}$$

What if the denominator doesn't factor completely into linear factors?

Ex $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$ Factor: $x^3+4x = x(x^2+4)$

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$\times$ by $x(x^2+4)$ $\quad 2x^2-x+4 = A(x^2+4) + (Bx+C)\,x$

method 1: plug in $x=0 \rightarrow 4 = 4A \quad A=1$

$x=1 \longrightarrow 5 = 5A+B+C \longrightarrow B+C=0$

$x=-1 \longrightarrow 7 = 5A+B-C \longrightarrow B-C=2$

$\longrightarrow$ $A=1,\ B=1,\ C=-1$

method ②: multiply out

$$2x^2-x+4 = Ax^2+4A+Bx^2+Cx$$

$$2x^2-x+4 = (A+B)x^2+Cx+4A$$

$$\left.\begin{array}{l} \rightarrow 2=A+B \\ -1=C \\ 4=4A \end{array}\right\} \longrightarrow \underline{A=1,\ C=-1,\ B=1}$$

So, $\displaystyle \int = \int \frac{A}{x} + \frac{Bx+C}{x^2+4}\,dx$

$$= \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx$$

$$= \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

$\ln|x|$

use $u=x^2+4$
get $\frac{1}{2}\ln|x^2+4|$

use $u=\frac{x}{2}$ $\quad x=2u$
get $-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

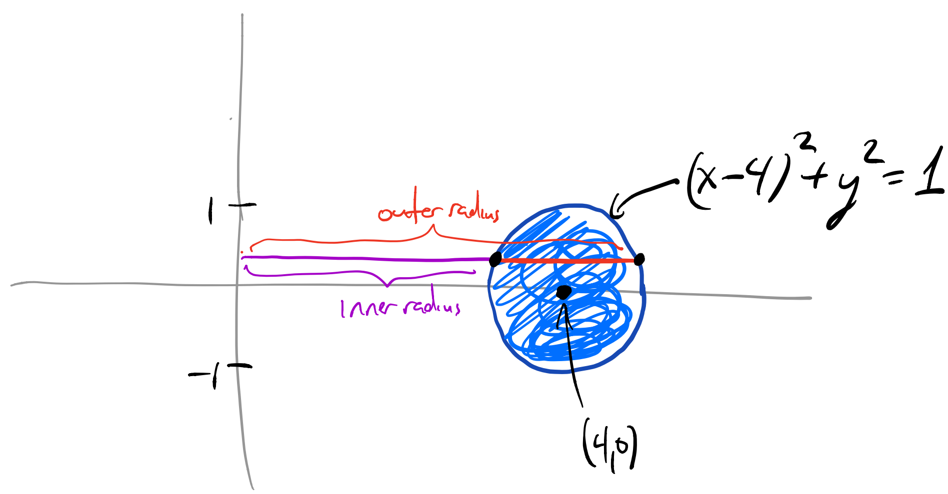
$$= \underline{\underline{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + K}}$$

$$\int \tan x \, dx = \ln|\sec x| + C$$
$$= -\ln|\cos x| + C$$

$$\int \frac{\sin x}{\cos x} dx \qquad u = \cos x, \quad du = -\sin x \, dx$$

$$\int -\frac{du}{u} = -\ln|u| = -\ln|\cos x| + C$$

Donut:



revolve around y-axis:

$$(x-4)^2 = 1 - y^2$$
$$x - 4 = \pm\sqrt{1-y^2}$$
$$x = 4 \pm \sqrt{1-y^2}$$

$$A = \int_{-1}^{1} \pi \left((\text{outer rad})^2 - (\text{inner rad})^2\right) dy$$

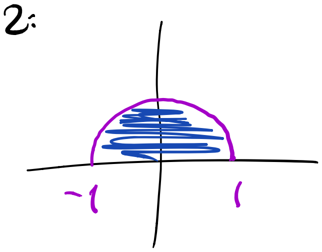
$$= \pi \int_{-1}^{1} \left(4+\sqrt{1-y^2}\right)^2 - \left(4-\sqrt{1-y^2}\right)^2 dy$$

$$= \pi \int_{-1}^{1} \left(16 + 8\sqrt{1-y^2} + (1-y^2)\right) - \left(16 - 8\sqrt{1-y^2} + (1-y^2)\right) dy$$

$$= \pi \int_{-1}^{1} 16\sqrt{1-y^2}\, dy$$

method 1: trig sub $y = \sin\theta$

method 2:



area $= \pi/2$

$$= \pi \cdot \left(16 \cdot \frac{\pi}{2}\right)$$

$$= \underline{\underline{8\pi^2}}$$

Rules of logs

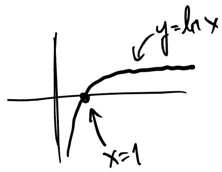
$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(x^a) = a \ln x$$

ex $\ln 3x^2$ $= \ln 3 + \ln x^2$ $= \ln 3 + 2\ln x$

$$\rightarrow \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

$$\ln\left(\frac{1}{b}\right) = \ln(b^{-1}) = -\ln b$$



$$\ln(e^x) = x \qquad e^{\ln x} = x \qquad \ln 1 = 0$$

$$\sin^{-1}(\sin x) = x$$

$$\int \tan^{-1} x\, dx$$

$u = \tan^{-1} x$ $\quad v = x$

$du = \dfrac{dx}{1+x^2}$ $\quad dv = dx$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2|$$

$\int \sin^{-1} x \, dx$ similarly

$$F(x) = \int_0^{x^{1/2}} \frac{16e^{-t^2}}{2+t^2} \, dt \qquad \text{find } F'(4)$$

FTC 1: $\frac{d}{dx} F(x) = \frac{d}{dx}(x^{1/2}) \cdot \frac{16e^{-x}}{2+x}$

$$= \frac{1}{2\sqrt{x}} \cdot \frac{16e^{-x}}{2+x} \qquad x=4$$

$$\frac{d}{dx}\left(\int_{x^2}^{x^3} \sin t \, dt\right) = \frac{d}{dx}\int_0^{x^3} \sin t \, dt + \frac{d}{dx}\int_{x^2}^{0} \sin t \, dt = 3x^2 \sin x^3 - 2x \sin x^2$$