Lecture 11

Exam 1 average in this section: 70%

across all 40?? : 68%

Partial fractions (con't)

To compute $\int \frac{P(x)}{Q(x)}\,dx$ $\quad P, Q$ polynomials

strategy: factor $Q(x)$, split up $\frac{P}{Q}$ into simpler pieces

Q $\displaystyle \int \frac{1}{x^2+x}\,dx = \int \frac{A}{x} + \frac{B}{x+1}\,dx$

$Q(x) = x(x+1)$

$$1 = A(x+1) + Bx$$

$$\left[\begin{array}{lll} x=0: & 1 = A & A=1 \\ x=-1: & 1 = -B & B=-1 \end{array}\right]$$

OR:

$$\left(\begin{array}{l} 1 = (A+B)x + A \\ 0x+1 = (A+B)x + A \\ \text{So: } 0 = A+B \\ \qquad 1 = A \\ \rightarrow B=-1,\ A=1 \end{array}\right)$$

$$\int \frac{1}{x}\,dx + \int \frac{-1}{x+1}\,dx$$

$$\underline{\ln|x| - \ln|x+1| + C} = \underline{\ln \frac{|x|}{|x+1|} + C}$$

Last time we did:

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$x(2x-1)(x+2)$

and

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$x(x^2+4)$

What if the degree of the numerator is $\geq$ the degree of denominator?

Ex $\displaystyle\int_0^1 \frac{x^3-4x+10}{x^2-x-6}\,dx = ?$

Do polynomial division:

$$\begin{array}{r|l} & x + 1 \\ x^2-x-6 & x^3+0x^2-4x+10 \\ & x^3-x^2-6x \\ \hline & x^2+2x+10 \\ & x^2-x-6 \\ \hline & 3x+16 \end{array}$$

$\rightarrow \dfrac{x^3-4x+10}{x^2-x-6} = x+1 + \dfrac{3x+16}{x^2-x-6}$

then $\displaystyle\int_0^1 \frac{x^3-4x+10}{x^2-x-6}\,dx = \int_0^1 x+1\,dx + \int_0^1 \frac{3x+16}{x^2-x-6}\,dx$

$$\frac{3x+16}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$A=5,\ B=-2$$

(check: $5(x+2)-2(x-3) = 3x+16$ ✓)

$$\int_0^1 x+1\,dx + \int_0^1 \frac{5}{x-3} + \frac{-2}{x+2}\,dx$$

$$\tfrac{1}{2}x^2+x\,\Big|_0^1 + 5\ln|x-3|\,\Big|_0^1 - 2\ln|x+2|\,\Big|_0^1$$

$$= \tfrac{3}{2} + 5\ln 2 - 5\ln 3 - 2\ln 3 + 2\ln 2$$

$$= \tfrac{3}{2} + 7\ln 2 - 7\ln 3 = \underline{\underline{\tfrac{3}{2} + 7\ln\tfrac{2}{3}}}$$

What if some factor appears <u>more than</u> <u>once</u> in the denominator?

<u>Ex</u> $\displaystyle\int \frac{1}{x^3+2x^2+x}\,dx$ Factor: x^3+2x^2+x

$$= x(x^2+2x+1)$$

$$= x(x+1)^2$$

write $$\frac{1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

[could also put $\frac{A}{x} + \frac{Bx+C}{(x+1)^2}$ — either will work]

$x(x+1)^2$

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$

$x=0 \rightarrow 1 = A$

$x=-1 \rightarrow 1 = -C \rightarrow C=-1$

$x=1 \rightarrow 1 = 4A+2B+C$

$1 = 3+2B \rightarrow B=-1$

then have $\displaystyle\int \frac{1}{x} + \frac{-1}{x+1} + \frac{-1}{(x+1)^2}\, dx$

$$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + K$$

Similarly, if $\dfrac{P(x)}{Q(x)}$ $\quad Q(x) = (x-3)^4(x+1)$

then we'd set

$$\frac{P(x)}{Q(x)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-3)^3} + \frac{E}{(x-3)^4}$$

solve for A, B, C, D, E

If $Q(x) = x^2(x+1)$ then do $\dfrac{P(x)}{Q(x)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

Q $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$ (repeated quadratic factor)

$$\rightarrow \frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$x(x^2+1)^2$

$$1-x+2x^2-x^3 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$= A(x^4+2x^2+1) + \underbrace{(Bx+C)(x^3+x)} + Dx^2+Ex$$

$$= Ax^4+2Ax^2+A + Bx^4+Bx^2+Cx^3+Cx + Dx^2+Ex$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\rightarrow \left.\begin{array}{l} A+B=0 \\ C=-1 \\ 2A+B+D=2 \\ C+E=-1 \\ A=1 \end{array}\right\} \quad \left.\begin{array}{l} A=1 \\ B=-1 \\ C=-1 \\ D=1 \\ E=0 \end{array}\right]$$

So our $\int$ is

$$\int \frac{1}{x} + \frac{(-x-1)}{x^2+1} + \frac{(x)}{(x^2+1)^2}\,dx$$

$\ln|x|$ $\quad -\frac{1}{2}\ln|x^2+1|$ $\quad u=x^2+1$

$-\tan^{-1}(x) + \frac{1}{2(x^2+1)}$

$$\rightarrow \ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$