$$\int \frac{1}{\sqrt{x^2+6x-7}}\,dx = ?$$

$$x^2+6x-7 = \underbrace{(x+3)^2}_{x^2+6x+9} - 16 = (4\sec\theta)^2 - 16$$

$$[x+3 = 4\sec\theta] \qquad = 16(\tan^2\theta)$$

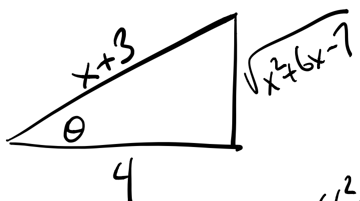
$$[dx = 4\sec\theta\tan\theta\,d\theta]$$

$$\int \frac{4\sec\theta\tan\theta}{\sqrt{16\tan^2\theta}}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\sec\theta \leftarrow \frac{x+3}{4} \qquad \tan\theta \leftarrow \frac{\sqrt{x^2+6x-7}}{4}$$

$x+3 = 4\sec\theta$

$\frac{x+3}{4} = \sec\theta$



$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{x+3}{4}$$

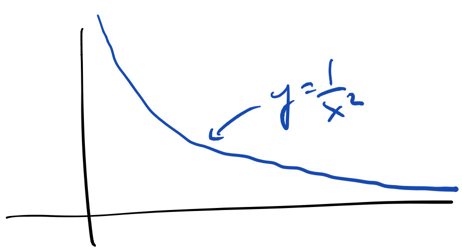
$$4^2 + c^2 = (x+3)^2$$

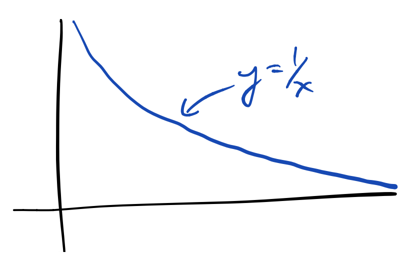
$$c = \sqrt{x^2+6x-7}$$

$$= \ln\left|\frac{x+3}{4} + \frac{\sqrt{x^2+6x-7}}{4}\right| + C$$

$$= \ln|x+3+\sqrt{x^2+6x-7}| - \ln 4 + C$$

Last time: <u>improper integrals</u> where the upper limit is $+\infty$





$$\int_1^\infty \frac{1}{x^2}\,dx$$
$$= \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^2}\,dx\right)$$
$$= \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$$

convergent

$$\int_1^\infty \frac{1}{x}\,dx$$
$$= \lim_{t\to\infty}\left(\int_1^t \frac{1}{x}\,dx\right)$$
$$= \lim_{t\to\infty} \ln t \quad \text{DNE (or, } = \infty)$$

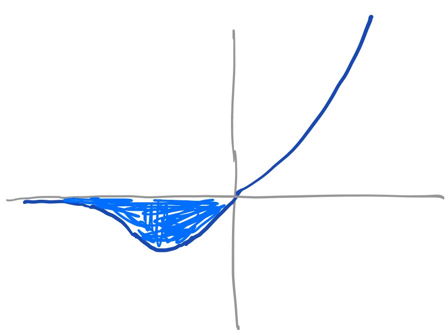
divergent

Slogan: $\frac{1}{x^2}$ goes to 0 <u>faster</u> than $\frac{1}{x}$ as $x\to\infty$

$\int_1^\infty \frac{1}{x^2}\,dx$ converges, $\int_1^\infty \frac{1}{x}\,dx$ doesn't.

General rule: for $a>0$, $\int_a^\infty \frac{1}{x^p}\,dx$ is $\begin{cases}\text{convergent if } p>1\\ \text{divergent if } p\le 1\end{cases}$

<u>Ex</u> $\int_{-\infty}^0 xe^x\,dx$ does it converge? if so, what is its value?



Need to do: $\lim_{t\to-\infty}\left(\int_t^0 xe^x\,dx\right)$

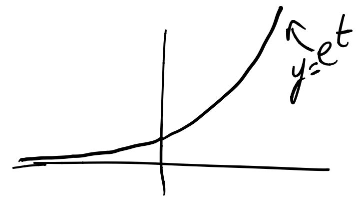
$$\int xe^x\,dx = xe^x - \int e^x\,dx$$
$u=x$ $\quad v=e^x$
$du=dx$ $\quad dv=e^x\,dx$

$$= xe^x - e^x$$

so $\int_t^0 xe^x\,dx = xe^x - e^x\Big|_t^0$

$$= -1 - te^t + e^t$$

$$\lim_{t\to-\infty} \left(\underset{-1}{-1} - \underset{\text{looks like } \infty\cdot 0}{te^t} + \underset{0}{e^t}\right)$$



use L'Hospital:

$te^t = \dfrac{t}{e^{-t}}$; as $t\to-\infty$ that's $\dfrac{-\infty}{\infty}$

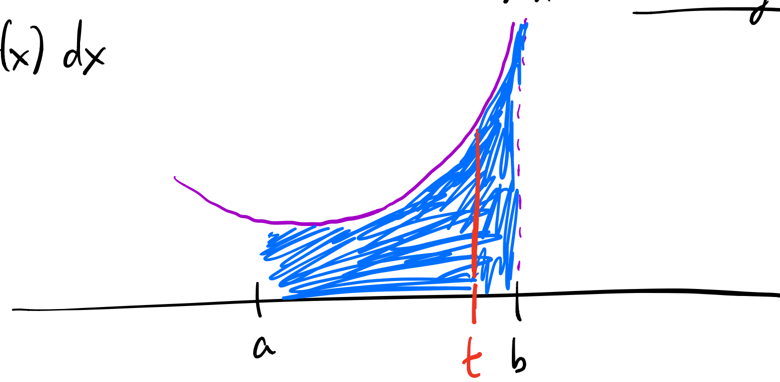
so use L'H rule to replace by $\displaystyle\lim_{t\to-\infty} \frac{1}{-e^{-t}} = \underline{\underline{0}}$.

So, finally get $\displaystyle\lim_{t\to-\infty} \left(-1-te^t+e^t\right) = -1+0+0 = \underline{\underline{-1}}$.

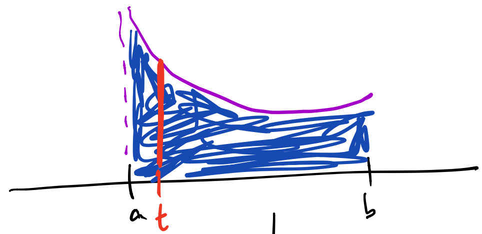
<u>Another</u> kind of improper integral: one where the <u>function</u> $f(x) \to \infty$.

ie $f(x)$ has a <u>vertical asymptote</u>

$$\int_a^b f(x)\,dx$$

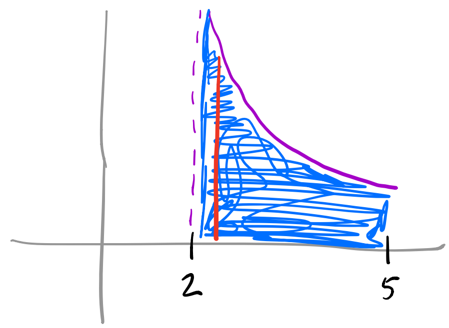


Here $\displaystyle\int_a^b f(x)\,dx$ <u>means</u> $\displaystyle\lim_{t\to b^-} \int_a^t f(x)\,dx$.



Here $\int_a^b f(x)\,dx$ means $\lim_{t\to a^+} \int_t^b f(x)\,dx$.

Q $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = ?$



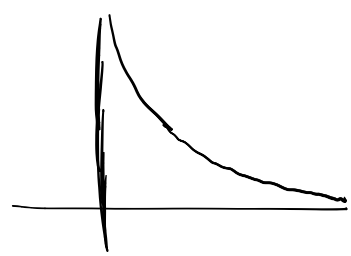
Improper because $\frac{1}{\sqrt{x-2}}$ has vert asymptote at $x=2$.

So, take $\displaystyle\lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$

$$= \lim_{t\to 2^+} \left(2\left(\sqrt{3}-\sqrt{t-2}\right)\right)$$

$$= 2\sqrt{3} \quad \text{(convergent)}.$$

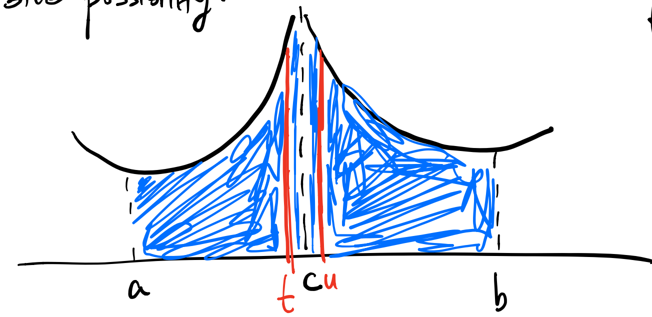
A general rule: $\displaystyle\int_0^a \frac{1}{x^p}\,dx$ is:



$$\begin{cases} \text{convergent if } p<1 \\ \text{divergent if } p \geq 1. \end{cases}$$

One other possibility:

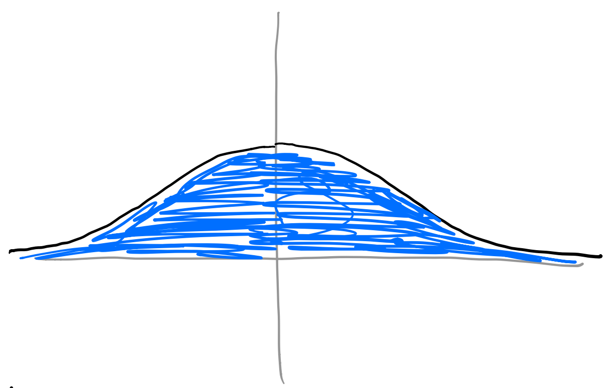
Here $\int_a^b f(x)\,dx$ means



$$\left(\lim_{t\to c^-}\int_a^t f(x)\,dx\right) + \left(\lim_{u\to c^+}\int_u^b f(x)\,dx\right)$$

Or:

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = ?$$



Define this by splitting up:

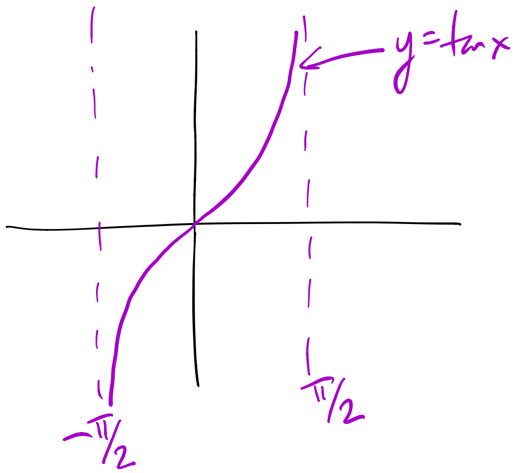
$$\int_0^{\infty}\frac{1}{1+x^2}\,dx + \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx$$

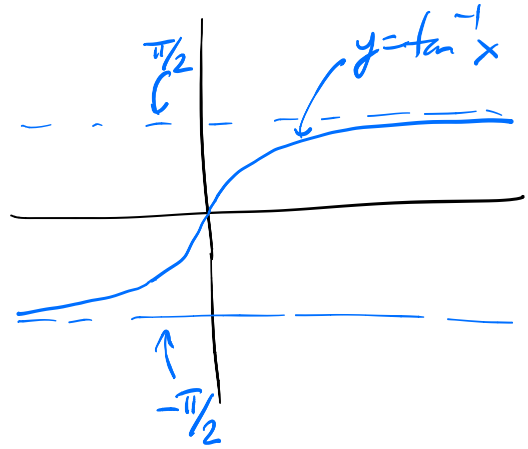
$$\lim_{t\to\infty}\left(\tan^{-1}x \Big|_0^t\right) + \lim_{t\to-\infty}\left(\tan^{-1}x \Big|_t^0\right)$$

$$= \lim_{t\to\infty}\left(\tan^{-1}t - \tan^{-1}0\right) + \lim_{t\to-\infty}\left(\tan^{-1}0 - \tan^{-1}t\right)$$

$$= \left(\frac{\pi}{2} - 0\right) + \left(0 - -\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \underline{\underline{\pi}}$$

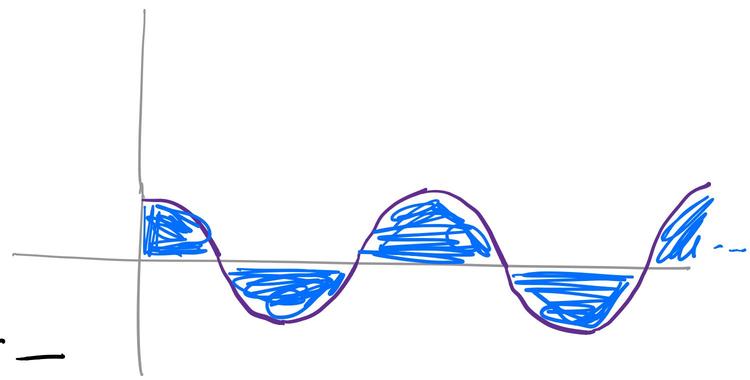




Q $\int_0^\infty \cos x \, dx$

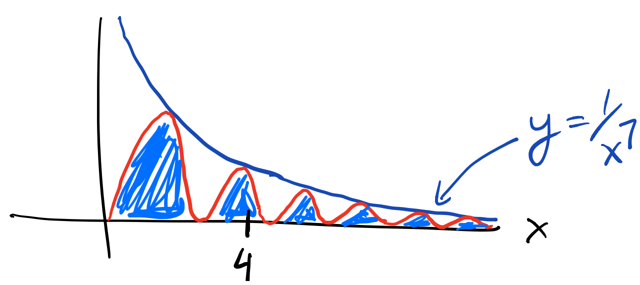
$= \lim_{t \to \infty} \sin t \Big|_0^\infty$

$= \lim_{t \to \infty} \sin t$ DNE — this integral diverges.



Comparison Theorem

Q Does $\int_4^\infty \frac{\sin^2 x}{x^7} \, dx$ converge?



$\int_4^\infty \frac{1}{x^7}\,dx$ converges, because $\int_4^\infty \frac{1}{x^p}$ conv. for $p>1$

And since $0 \leq \sin^2 x \leq 1$

$$0 \leq \frac{\sin^2 x}{x^7} \leq \frac{1}{x^7}$$

So $\int_4^\infty \frac{\sin^2 x}{x^7}\,dx < \int_4^\infty \frac{1}{x^7}\,dx$, so it converges