Imagine you finish a HW problem

$$\int (\text{stuff}) = \ln\left|\frac{x}{6}+\frac{\sqrt{x^2-7x+9}}{6}\right| + C$$

and QUEST shows $\ln\left|x+\sqrt{x^2-7x+9}\right| + C$

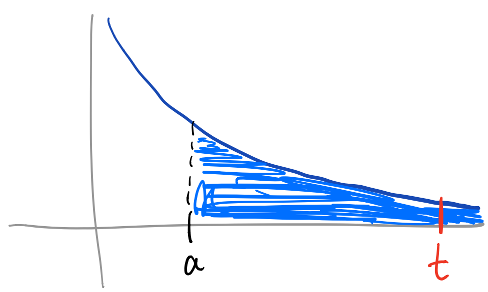
This is actually correct!

Because

$$\ln\left|\frac{x}{6}+\frac{\sqrt{x^2-7x+9}}{6}\right| + C$$

$$= \ln\left|x+\sqrt{x^2-7x+9}\right| + \underbrace{\ln\left(\frac{1}{6}\right) + C}_{\text{just a constant!}}$$

Last time: improper integrals $\int_a^\infty f(x)\,dx$



By definition,

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Last time we did: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1-\frac{1}{t} = 1$ convergent

$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = \infty$ (or DNE) divergent

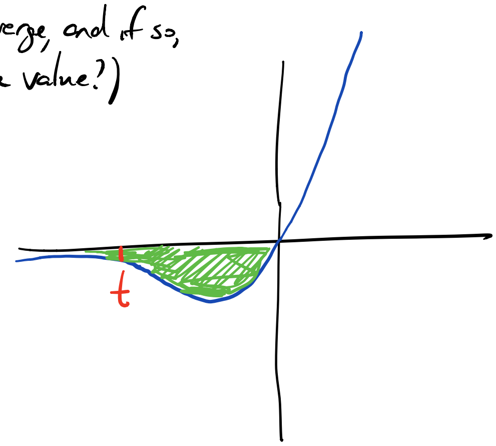
($\frac{1}{x^2}$ goes to 0 "faster" than $\frac{1}{x}$ does, as $x\to\infty$)

General rule: for $a>0$, $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ is <u>convergent</u> if $p>1$, <u>divergent</u> if $p\le 1$

<u>Ex</u> $\displaystyle\int_{1042}^\infty \frac{1}{x^3}\,dx$ is <u>convergent</u> $(3>1)$

$\displaystyle\int_{1/3}^\infty \frac{1}{\sqrt{x}}\,dx$ is <u>divergent</u> $(\tfrac{1}{2}<1)$

<u>Q</u> $\displaystyle\int_{-\infty}^0 xe^x\,dx = ?$ (does it converge, and if so, what is the value?)



We define this improper $\int$ as

$$\lim_{t\to-\infty}\left(\int_t^0 xe^x\,dx\right)$$

IBP: $u=x$ $\quad v=e^x$
$du=dx$ $\quad dv=e^x\,dx$

$$xe^x\Big|_t^0 - \int_t^0 e^x\,dx$$

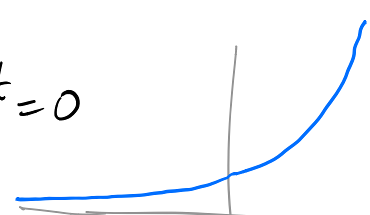
$$= -te^t - (1-e^t)$$

$$= -te^t - 1 + e^t$$

$$\lim_{t\to-\infty}\left(\underset{\substack{\uparrow\\ \infty\cdot 0}}{-te^t} - 1 + e^t\right)$$

$= \quad$ need to use L'H rule! $\quad -1+0$

$$\lim_{t\to-\infty} e^t = 0$$



$$e^{-100} = \frac{1}{e^{100}} \approx \frac{1}{3^{100}} \quad \underline{\underline{\text{small}}}$$

$$\lim_{t\to-\infty} -\frac{t}{e^{-t}} \quad \text{looks like } \frac{+\infty}{+\infty}$$

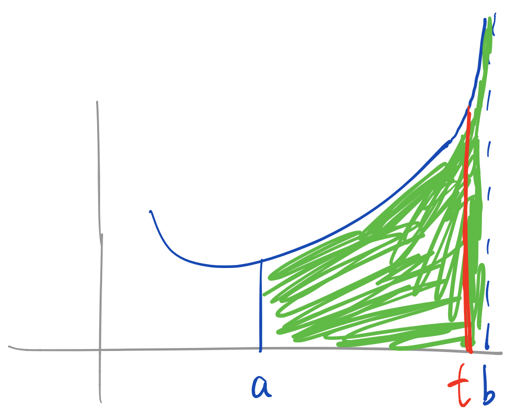
$$\overset{L'H}{=} \lim_{t\to-\infty} \frac{-1}{-e^{-t}} \longrightarrow \frac{1}{\infty} = \underline{0}$$

So finally $\lim_{t\to-\infty}\left(-te^{t}-1+e^{t}\right) = 0-1+0 = \underline{-1}$

(convergent)

Another kind of improper $\int$: $\int_a^b f(x)\,dx$ where $f(x)$ becomes ∞ for some x in $[a,b]$.

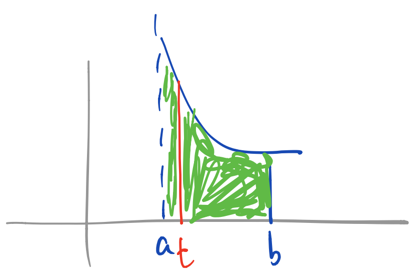
(ie $f(x)$ has a vertical asymptote)



Here $\int_a^b f(x)\,dx$ means

$$\lim_{t\to b^-} \int_a^t f(x)\,dx$$

Similarly for

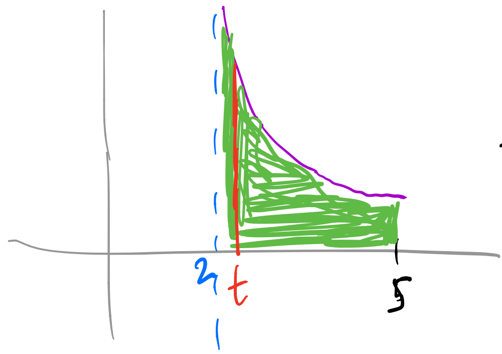


here $\int_a^b f(x)\,dx$ means

$$\lim_{t\to a^+} \int_t^b f(x)\,dx$$

Q $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ does it converge? if so, to what?

Q $\displaystyle\int_{-2}^3 \frac{1}{x}\,dx$?



$$\lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t\to 2^+} \left(2\sqrt{x-2}\,\Big|_t^5 \right)$$

$$= \lim_{t\to 2^+} \left(2\sqrt{3} - 2\sqrt{t-2} \right)$$

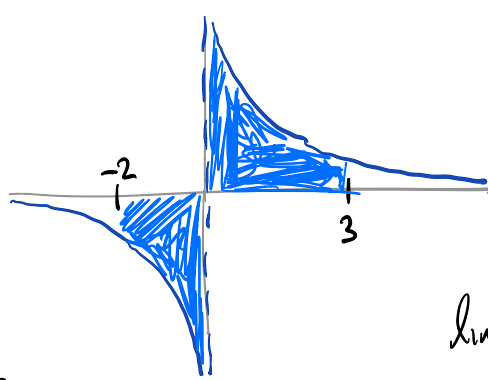
$$= 2\sqrt{3} - 0$$

$$= \underline{\underline{2\sqrt{3}}} \quad \text{(convergent)}$$

A general rule: (for $a>0$) $\displaystyle\int_0^a \frac{1}{x^p}$ is $\begin{cases} \text{convergent} & p<1 \\ \text{divergent} & p\geq 1 \end{cases}$

(same for $\displaystyle\int_c^a \frac{1}{(x-c)^p}$) (we just did example $p=\frac{1}{2}$)

For $\displaystyle\int_{-2}^3 \frac{1}{x}\,dx$:



$\underline{\underline{\text{vertical asymp at } x=0!}}$

We define it to be

$$\lim \int_{-2}^t + \int_{-t}^3$$

$$\int_{-2}^0 \frac{1}{x}\,dx + \int_0^3 \frac{1}{x}\,dx$$

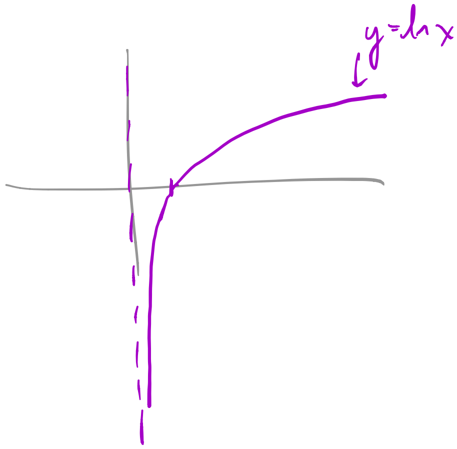
$\underline{\text{both divergent}} \implies$ this $\int$ is divergent.

(using our rule, or explicitly $\displaystyle\int_{-2}^{0} \frac{1}{x}\,dx = \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x}\,dx$

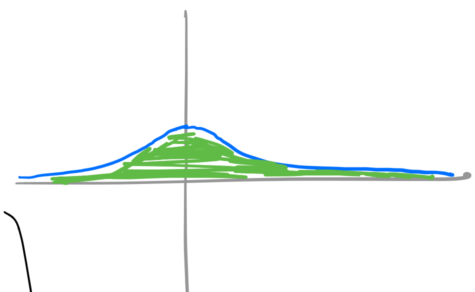
$$= \lim_{t\to 0^-} \ln|x| \Big|_{-2}^{t}$$

$$= \lim_{t\to 0^-} \left(\ln|t| - \ln 2\right)$$

DNE)



Q $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = ?$



(We define this $\int$ by splitting it up:

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx + \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx \ )$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx + \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$$

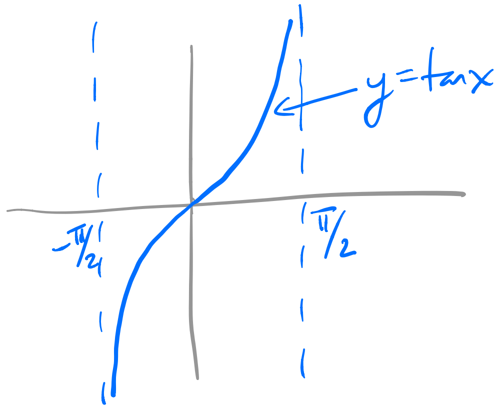
$$= \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx + \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx$$

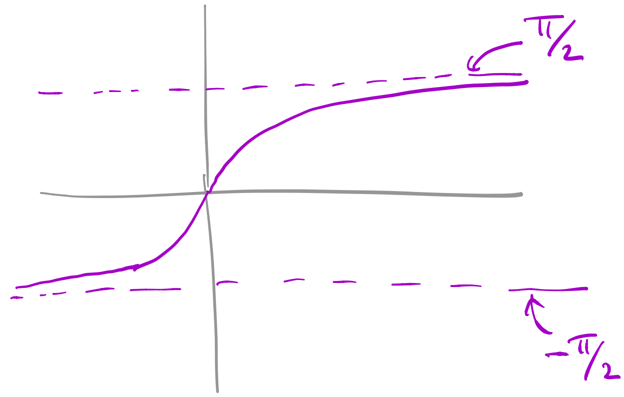
$$= \lim_{t\to\infty} \left(\tan^{-1}x \Big|_{0}^{t}\right) + \lim_{t\to-\infty} \left(\tan^{-1}x \Big|_{t}^{0}\right)$$

$$= \lim_{t\to\infty} \tan^{-1}t - \lim_{t\to-\infty} \tan^{-1}t$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \underline{\underline{\pi}}$$

so $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \underline{\underline{\pi}}$



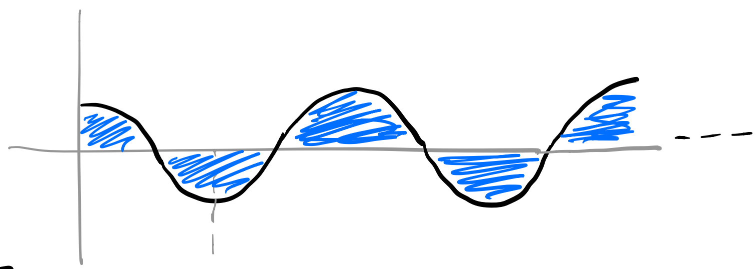


Ex $\int_0^{\infty} \cos x \, dx$

$$= \lim_{t\to\infty} \sin t$$

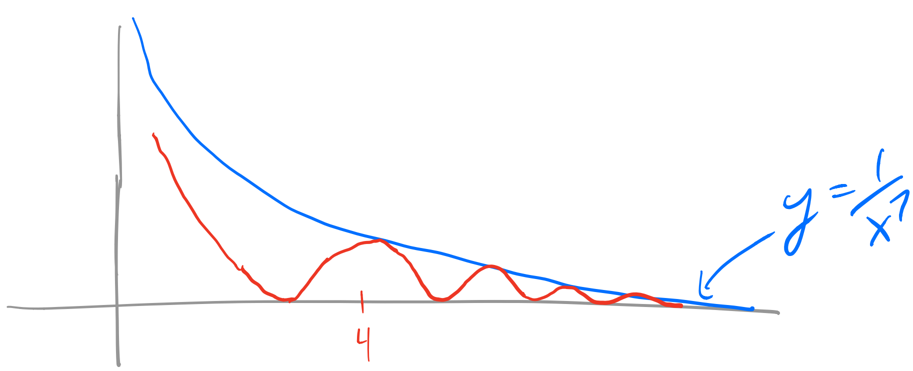
DNE

so this $\int$ is <u>divergent</u>.



<u>Comparison Theorem</u>

<u>Q</u> Does $\int_4^{\infty} \frac{\sin^2(x)}{x^7}\,dx$ <u>converge</u>?



Yes, becaus $\int_4^\infty \frac{1}{x^7}\,dx$ converges (p-test)

and $0 < \frac{\sin^2 x}{x^7} < \frac{1}{x^7}$