Lecture 14

Exam 2 Tue Oct 31 7–9pm

Last time: improper integrals

So far, always looked at functions $f(x)$

In reality things often depend on >1 variable —

e.g. heat index depends on temperature and humidity $f(x,y)$

price of cheese depends on supply and demand

productivity — supply of labor and supply of capital

Partial derivatives

If we have $f(x,y)$ we ask:

- How does $f(x,y)$ change when we vary x and hold y fixed?

Define $$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$ "partial deriv with respect to x"

To calculate $\frac{\partial f}{\partial x}$, treat y as a constant and x as a variable

Similarly, To calculate $\frac{\partial f}{\partial y}$, treat x as a constant and y as a variable

Q If $f(x,y) = x^2 \sin(y)$ what is $\frac{\partial f}{\partial x}$? what is $\frac{\partial f}{\partial y}$?

what is $\frac{\partial f}{\partial x}$ at $(x,y) = (1, \pi/2)$?

A $f(x,y) = x^2 \sin(y)$

$$\frac{\partial f}{\partial x} = \underline{2x\sin(y)} \qquad \frac{\partial f}{\partial x} \text{ at } (x,y) = (1, \pi/2): \; 2(1)\sin(\pi/2) = \underline{2}$$

$$\frac{\partial f}{\partial y} = \underline{x^2\cos(y)}$$

Can also take <u>second derivatives</u>:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

<u>Q</u> For $f(x,y) = 4x^2y + 7\sin(x) + 8$

what are all the partial derivs. and 2^{nd} partial derivs?

$$\frac{\partial f}{\partial x} = \underline{8xy + 7\cos(x)}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(8xy + 7\cos x) = \underline{8y - 7\sin x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(8xy + 7\cos x) = \underline{8x}$$

$$\frac{\partial f}{\partial y} = \underline{4x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(4x^2) = \underline{8x}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(4x^2) = \underline{0}.$$

<u>Note</u>: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$! (Always true for functions which are nice enough — derivatives exist and are continuous)

Q Say $f(x,y) = \sin(xy)$.
Compute all the second derivatives.

Notation:
$$f_x = \frac{\partial f}{\partial x} \quad f_y = \frac{\partial f}{\partial y}$$
$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

$$f_x = y\cos(xy) \qquad f_y = x\cos(xy)$$

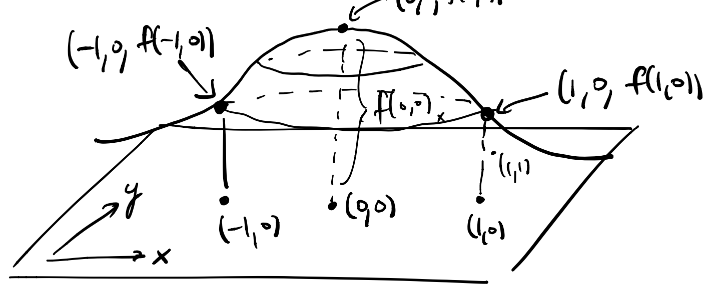
$$f_{xx} = y \cdot y \cdot (-\sin(xy)) = \underline{\underline{-y^2\sin(xy)}} \qquad f_{yy} = x \cdot x \cdot (-\sin(xy)) = \underline{\underline{-x^2\sin(xy)}}$$

$$f_{xy} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\left(x\cos(xy)\right) = \cos(xy) + x \cdot y \cdot (-\sin(xy)) = \underline{\underline{\cos(xy) - xy\sin(xy)}}$$

check: $f_{yx} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\left(y\cos(xy)\right) = \underline{\underline{\cos(xy) - yx\sin(xy)}}$

What's the meaning of partial derivatives?

If we have $f(x,y)$ we can make its graph $\underline{\underline{z = f(x,y)}}$



e.g. say $f(x,y) = e^{-x^2-y^2}$. What do partial deriv. mean?

Q Calculate f_x and f_y and $f_x(1,0)$ $f_y(1,0)$

$f_x(1,1)$ $f_y(1,1)$.

$$f_x = -2x e^{-x^2-y^2}$$
$$f_y = -2y e^{-x^2-y^2}$$

$$f_x(1,0) = -2(1)e^{-1} = -2/e < 0$$
$$f_y(1,0) = -2(0)e^{-1} = \underline{0}$$
$$f_x(1,1) = -2e^{-2} = -2/e^2 < 0$$
$$f_y(1,1) = -2e^{-2} = -2/e^2 < 0$$

Q $f_x(1,0) < 0$. What does this <u>mean</u> for the graph of f?

It means that f is <u>decreasing</u> as we <u>increase</u> x, with y <u>fixed</u>.

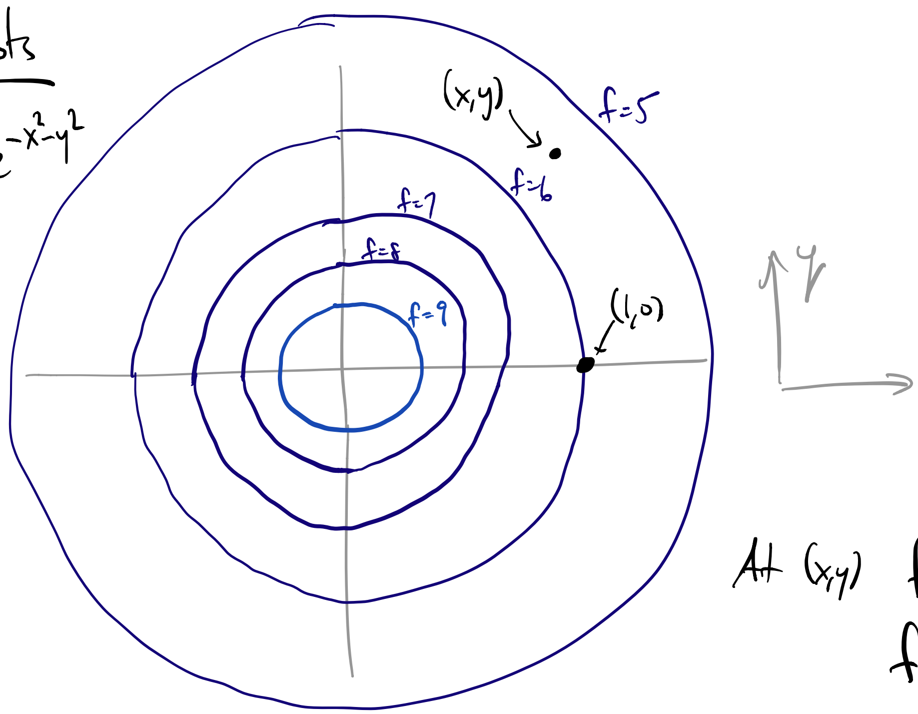
Q $f_y(1,0) = 0$. What does this mean for the graph of f?

Means that <u>with x fixed</u>, $y=0$ is a <u>critical point</u> for $f(1,y)$.

(in fact, local max)

Contour plots

$f(x,y) = 10e^{-x^2-y^2}$



At (x,y) $f_x < 0$

$f_y < 0$

At $(1,0)$ $f_x < 0$

$f_y = 0$