Sequences

A **sequence** is an ordered list of numbers

$$\{a_n\} = a_1, a_2, a_3, a_4, \dots, a_{100}, \dots$$

Ex $a_n = n$: $1, 2, 3, 4, \dots, 100, \dots$

$a_n = n^2$: $1, 4, 9, 16, \dots, 10000, \dots$

$a_n = (-1)^n$: $-1, 1, -1, 1, \dots, 1, \dots$

$a_n = \frac{1}{n^2+1}$: $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \dots, \frac{1}{10001}, \dots$

a_n = the price of 1 Bitcoin in USD at midnight on day n of 2017:
4211.04, 4317.19, ..., 6034.88, ...

a_n = the nth term of the Fibonacci sequence (ie $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$)
$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$

$a_n = \frac{1}{n}\cos\left(\frac{n\pi}{2}\right)$
$0, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{6}, 0, \frac{1}{8}, 0, -\frac{1}{10}, \dots$

$a_n = n!$ $\quad (n! = n(n-1)(n-2)\cdots(2)(1))$

$1, \underset{2\cdot1}{2}, \underset{3\cdot2\cdot1}{6}, \underset{4\cdot3\cdot2\cdot1}{24}, 120, 720, 5040, \dots$ eg $5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$

Q Consider the sequence $\underset{a_1}{3}, \underset{a_2}{8}, \underset{a_3}{13}, 18, 23, \dots$
where each term differs from the previous one by 5.
What is a formula for a_n?

one answer: $a_n = a_{n-1} + 5, \quad a_1 = 3$

or: $a_n = 5n - 2$

<u>Motivation</u> for sequences: we can use them to calculate things we actually care about —

e.g. if we want to calculate $\sin(x)$

we take a sequence of <u>approximations</u> to $\sin(x)$

e.g. if x is small, like $x \approx 0.01$, $\sin(x) \approx 0$

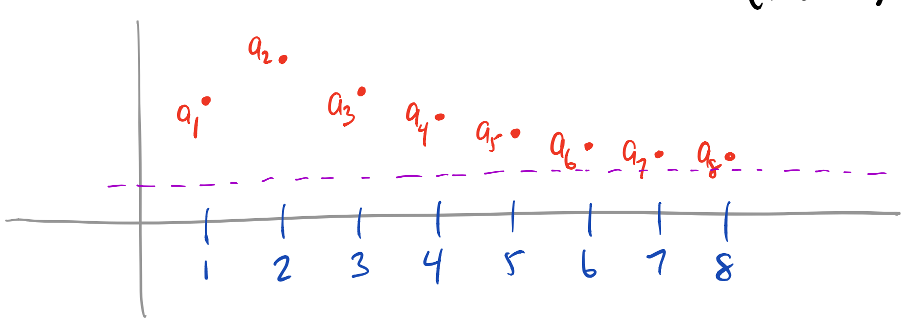
<u>better</u>, $\sin(x) \approx 0.01$

<u>better</u>, $\sin(x) \approx 0.01 + \frac{(0.01)^3}{6} \dots$

<u>sequence</u>:

$$x, \quad x + \frac{x^3}{6}, \quad x + \frac{x^3}{6} + \frac{x^5}{120}, \quad \dots$$

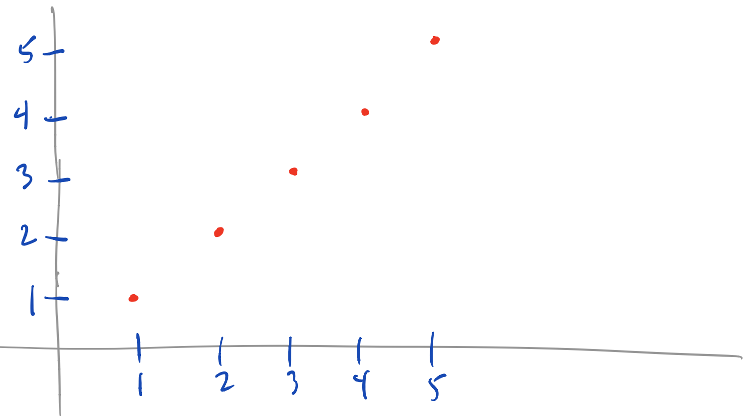
Fundamental question about a sequence $\{a_n\}$: does it <u>converge</u>? (to what?)



the seq. <u>converges</u> if (informally) the graph of the seq. approaches a <u>horizontal asymptote</u>

and <u>diverges</u> if not

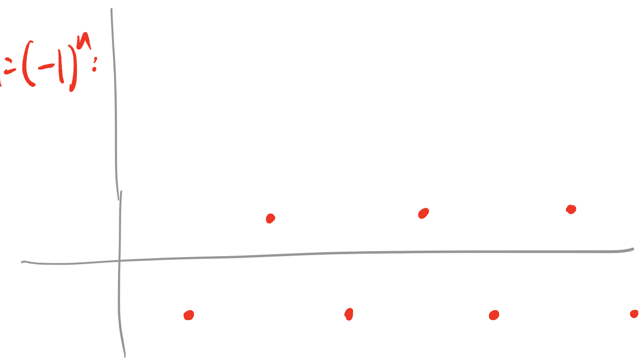
$a_n = n$:



$1, 2, 3, 4, 5, \ldots$

diverges

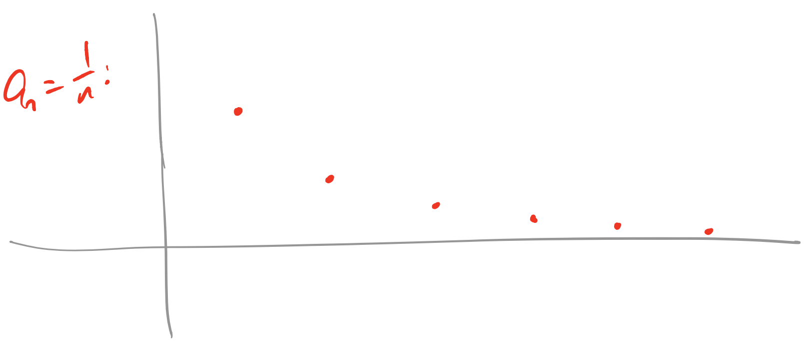
$a_n = (-1)^n$:



$-1, 1, -1, 1, -1, \ldots$

diverges

$a_n = \frac{1}{n}$:



$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

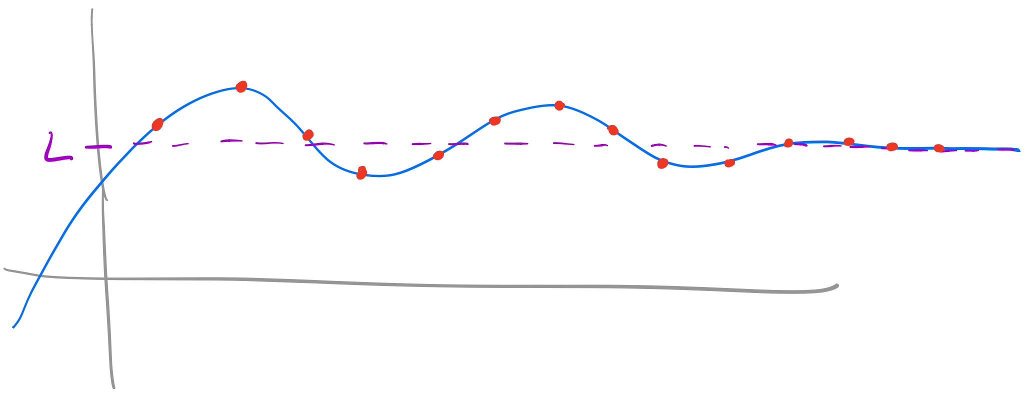
converges to 0

We also write this as $\lim_{n \to \infty} a_n = 0$, i.e. $\lim_{n \to \infty} \frac{1}{n} = 0$.

This might remind you of the fact that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Indeed: if our sequence is given by a function $a_n = f(n)$

and $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} a_n = L$



Exam: 16 problems, roughly as hard as the last one

Don't need to know "product to sum" $\sin(4x)\cos(7x) = \cdots$

Should know

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta) \qquad \sin^2 + \cos^2 = 1$$

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta) \qquad \tan^2 + 1 = \sec^2$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta \, d\theta = -\ln|\csc\theta + \cot\theta| + C$$

$$= \ln|\csc\theta - \cot\theta| + C$$

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx \qquad x = 3\sec\theta$$

$$dx = 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}$$

$$= \int \frac{1}{3}\frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\cdot 3\tan\theta} = \frac{1}{9}\int\frac{1}{\sec\theta}\,d\theta$$

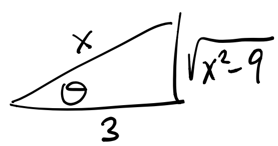
$$= \frac{1}{9}\int\cos\theta\,d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

$\sec\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{x}$



$\sin\theta = \frac{\sqrt{x^2-9}}{x}$

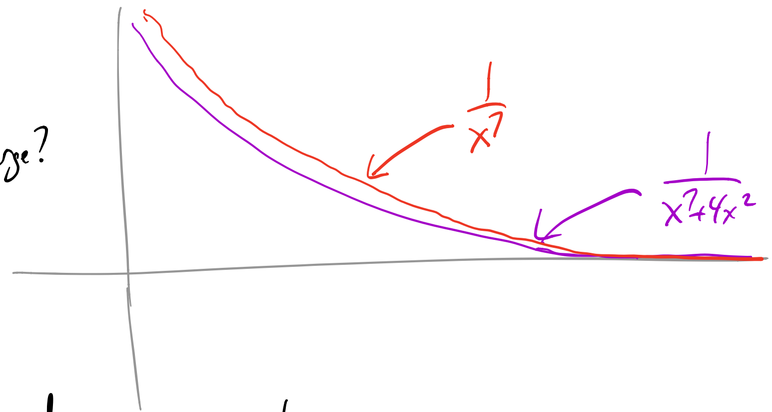
$$\int \frac{3x^2}{\sqrt{x^6-9}}\,dx \qquad u = x^3$$

$$\int \tan x\sec^2 x\,dx \qquad u =$$

get $\frac{\tan^2 x}{2} + C$ $\qquad \frac{\sec^2 x}{2} + C$ $\qquad \tan^2 = \sec^2 - 1$

Comparison test:

Does $\displaystyle\int_1^\infty \frac{dx}{x^7+4x^2}$ converge?



it's smaller than

$\displaystyle\int_1^\infty \frac{dx}{x^7}$ which does converge, so it converges

$$\int \frac{\cos^2 x}{x^4+10x^3}\,dx$$

$$\int_1^2 \frac{\sqrt{x-1}}{x}\,dx$$

$u=\sqrt{x-1}$

$u^2=x-1$

$2u\,du=dx$

$$=\int_0^1 \frac{u\cdot 2u\,du}{u^2+1}$$

$$=\int_0^1 \frac{2u^2}{u^2+1}\,du$$

$$=\int_0^1 2-\frac{2}{u^2+1}\,du$$

$$=2u-2\tan^{-1}u\,\Big|_0^1 = \underline{\underline{2-\frac{\pi}{2}}}$$

$$\int \frac{2}{u^2+4}\,du$$

$u = 2t$
$du = 2dt$

$$= \int \frac{2 \cdot 2dt}{4t^2+4}$$

$$= \int \frac{dt}{t^2+1} = \tan^{-1} t = \tan^{-1}\left(\frac{u}{2}\right)$$

$$\int \sin^{-1} 9x \, dx$$

$u = \sin^{-1} 9x$ $\quad v = x$

$du = \frac{9}{\sqrt{1-(9x)^2}}dx$ $\quad dv = dx$

$$\shortparallel$$

$$x \sin^{-1} 9x - \int \frac{9x}{\sqrt{1-(9x)^2}}\,dx$$

$u = 1 - 81x^2$
$du = -162x \, dx$
$-\frac{du}{162} = x\,dx$

$$-\int \frac{9\left(-\frac{du}{162}\right)}{\sqrt{u}}$$

$$+\frac{9}{162}\int \frac{du}{\sqrt{u}}$$

$$\frac{1}{18} \cdot 2u^{1/2} = \frac{1}{9}\sqrt{u}$$

$$\underline{x \sin^{-1} 9x + \frac{1}{9}\sqrt{1-81x^2}}$$

$$\int_0^3 \left(\frac{9}{x^2+9}\right)^2 dx$$

$x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

$$81\int_0^{\pi/4} \frac{3\sec^2\theta}{(9\tan^2\theta+9)^2} d\theta$$

$$= \cancel{81}\int_0^{\pi/4} \frac{3\sec^2\theta}{\cancel{81}\sec^4\theta}$$

$$= 3\int_0^{\pi/4} \cos^2\theta \, d\theta$$

...

$$\int \frac{x^3}{x^2-4} dx$$

$$= \int x + \frac{4x}{x^2-4} dx$$

$$\begin{array}{r|l} & x \\ x^2-4 & x^3+0x^2+0x+0 \\ & -(x^3 \quad -4x) \\ \hline & 4x \end{array}$$

$= \ldots$

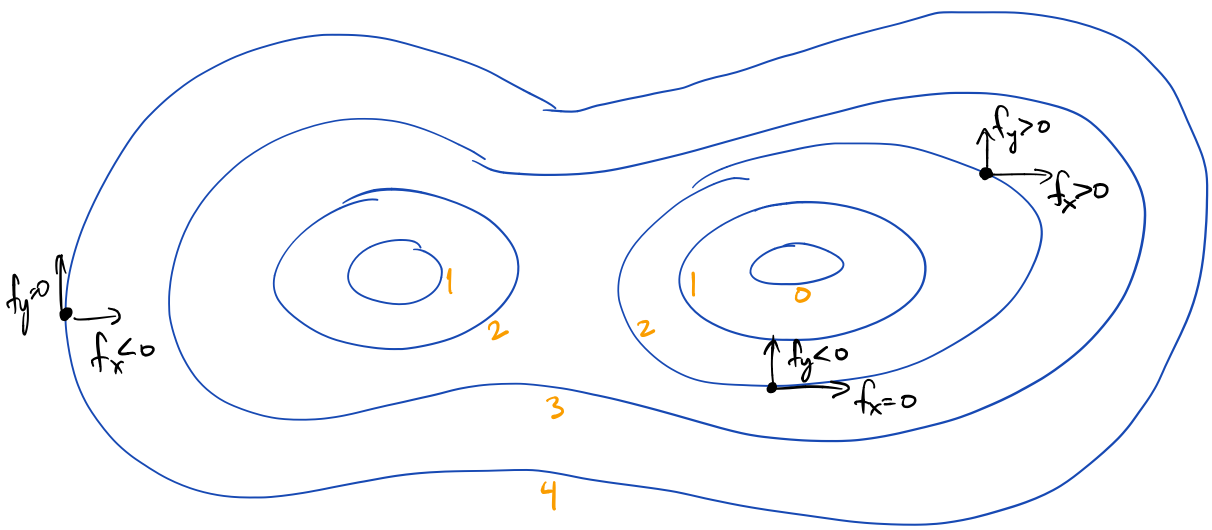
$$\int \frac{-x^2+3x+4}{x^3+x} dx$$

$$\frac{-x^2+3x+4}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$-x^2+3x+4 = A(x^2+1)+(Bx+C)x$$

$$-x^2+3x+4 = Ax^2+A+Bx^2+Cx$$
$$= (A+B)x^2+Cx+A$$

$$-1 = A+B$$
$$3 = C \qquad \ldots$$
$$4 = A$$



$$\int_0^{\frac{1}{\sqrt2}} \sqrt{\frac{1+x}{1-x}}\,dx = \int_0^{\frac{1}{\sqrt2}} \sqrt{\frac{1+x}{1-x}}\cdot\sqrt{\frac{1+x}{1+x}}\,dx$$
$$= \int_0^{\frac{1}{\sqrt2}} \frac{1+x}{\sqrt{1-x^2}}\,dx$$
$$= \int_0^{\frac{1}{\sqrt2}} \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^{\frac{1}{\sqrt2}} \frac{x}{\sqrt{1-x^2}}\,dx$$
$$= \sin^{-1}x \Big|_0^{\frac{1}{\sqrt2}} + \int_1^{\frac12} \frac{-\frac12\,du}{\sqrt u} \qquad \begin{array}{l} u=1-x^2 \\ du=-2x\,dx \end{array}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \left. -u^{1/2} \right|_{1}^{1/2}$$

$$= \pi/4 + \left(-\frac{1}{\sqrt{2}} + 1\right)$$

$$= \pi/4 + 1 - \frac{1}{\sqrt{2}}$$