Exam 2 results: this class average 65%
all M408L avg 62%

Sequences

Last time: sequence is an ordered list of numbers a_n

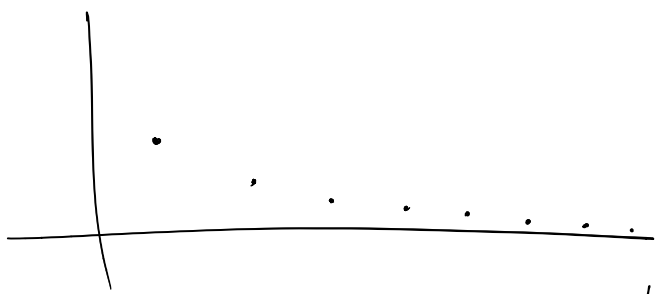
e.g. $a_n = \frac{\ln n}{n}$: $0, \frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4}, \ldots$

Q 1) Does the sequence $a_n = \frac{\ln n}{n}$ converge, and if so to what?

2) ——— $a_n = \frac{\ln n}{n} - 4$ — ——— ?

1) converges to zero:

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$



how to see this?

look at the function $f(x) = \frac{\ln x}{x}$ $\quad \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

2) $$\lim_{n\to\infty} \frac{\ln n}{n} - 4 = \left(\lim_{n\to\infty} \frac{\ln n}{n}\right) - \left(\lim_{n\to\infty} 4\right)$$
$$= 0 - 4 = -4$$

Q Does $a_n = \sin\left(\frac{\pi n}{1+4n}\right)$ converge?

$$\lim_{n\to\infty} \sin\left(\frac{\pi n}{1+4n}\right) = \lim_{n\to\infty} \sin\left(\frac{\pi}{\frac{1}{n}+4}\right)$$

$$= \sin\left(\lim_{n\to\infty} \frac{\pi}{\frac{1}{n}+4}\right)$$

$$= \sin\left(\frac{\pi}{4}\right) = \underline{\underline{\frac{\sqrt{2}}{2}}}$$

Q $\displaystyle\lim_{n\to\infty} 2+\frac{1}{n}+\frac{n!+1}{(n+1)!}$

$$\frac{n(n-1)(n-2)\cdots}{(n+1)\,n(n-1)(n-2)\cdots}$$

$$= \lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{n!}{(n+1)!} + \lim_{n\to\infty}\frac{1}{(n+1)!}$$

$$= 2 + 0 + \lim_{n\to\infty}\frac{1}{n+1} + 0$$

$$= 2 + 0 + 0 + 0$$

$$= \underline{\underline{2}}$$

Q Does the seq. $a_n = \left(1-\frac{7}{n}\right)^{-2n}$ converge? (to what?)

plugging in large values of n, $\sim 1^{-\infty}$

indeterminate form, like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $0\cdot\infty$

$\Rightarrow$ need to treat the whole thing together, not treat pieces separately

To get the limit, use:

$$\boxed{\text{Fact: } \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$$

(if $x=1$, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$)

$a_n = \left(1-\frac{7}{n}\right)^{-2n}$

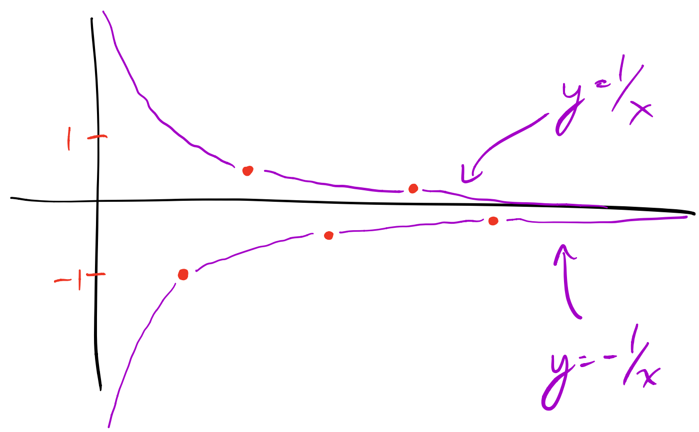
$$= \left[\left(1-\frac{7}{n}\right)^n\right]^{-2}$$

$$\lim_{n\to\infty} a_n = \left[\lim_{n\to\infty}\left(1+\frac{-7}{n}\right)^n\right]^{-2}$$

$$= \left(e^{-7}\right)^{-2} = \underline{\underline{e^{14}}}$$

Q $\lim_{n\to\infty} \frac{(-1)^n}{n} = ?$

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots$



$$\lim_{n\to\infty} \frac{(-1)^n}{n} = \underline{\underline{0}}$$

Fact: if you have a sequence a_n where $\lim_{n\to\infty} |a_n| = 0$

then $\lim_{n\to\infty} a_n = 0$.

(like in the example above: $|a_n| = \frac{1}{n}$

and $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} a_n = 0$ also.)

NB: $a_n = \frac{(-1)^n}{n} = (-1)^n \cdot \frac{1}{n}$

converges   diverges   converges

But,

$$a_n = n^2 = n^3 \cdot \frac{1}{n}$$

diverges diverges converges

Lesson: a product (diverges) $\times$ (converges) might converge or diverge

But, it is OK to say (converges) $\times$ (converges) = (converges)

Ex
$$\lim_{n\to\infty} \tan^{-1}(n) \cdot \frac{n^2+3}{4n^2+7}$$

$$= \left(\lim_{n\to\infty} \tan^{-1}(n)\right) \cdot \left(\lim_{n\to\infty} \frac{n^2+3}{4n^2+7}\right) \leftarrow \frac{1+\frac{3}{n^2}}{4+\frac{7}{n^2}}$$

$$= \left(\frac{\pi}{2}\right) \cdot \left(\frac{1}{4}\right)$$

$$= \underline{\underline{\frac{\pi}{8}}}$$

Series

Take a sequence a_n. Try to take the sum of all the terms in the sequence.

Ex $a_n = n$: $1+2+3+4+5+\cdots = \sum_{n=1}^{\infty} n$

Ex $a_n = \frac{1}{2^n}$: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$

(Recall summation notation:

e.g. $1+4+9+16+25+36$

$= 1^2+2^2+3^2+4^2+5^2+6^2$

$= \sum_{i=1}^{6} i^2$)

How do we understand this question?

Like we did for improper integrals $\int^{\infty}$:

look at the sum of the first k terms $\quad S_k = \sum_{n=1}^{k} a_n$ "partial sum"

Then try to take limit as $k \to \infty$.

$\lim_{k\to\infty} S_k$

If that exists, say the series converges, otherwise it diverges.

Ex if $a_n = n$:

1, 2, 3, 4, 5, ...

$S_1 = 1 \qquad\qquad = 1$

$S_2 = 1+2 \qquad\quad = 3$

$S_3 = 1+2+3 \qquad = 6$

$S_4 = 1+2+3+4 \quad = 10$

$\vdots \qquad\qquad\qquad \vdots$

$\{S_n\}$ 1, 3, 6, 10, ...

diverges

So, $\sum_{n=1}^{\infty} n$ diverges

Ex if $a_n = \frac{1}{2^n}$:

$S_1 = \frac{1}{2} \qquad\qquad\qquad = \frac{1}{2}$

$S_2 = \frac{1}{2}+\frac{1}{4} \qquad\qquad = \frac{3}{4}$

$S_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} \qquad = \frac{7}{8}$

$S_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16}$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$\lim_{n\to\infty} S_n = 1$

So, $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

NB: If we know the partial sums s_n we can recover the
(nota bene) original sequence a_n by

$$a_n = S_n - S_{n-1}$$

Basic example: geometric series

e.g. $a_n = 2, 6, 18, 54, 162, 486, \dots$

$\times 3 \quad \times 3 \quad \times 3 \quad \times 3 \quad \times 3$

$$a_n = 2 \cdot 3^{n-1}$$

(or in general $a_n = a \cdot r^{n-1}$)

1st term ratio between successive terms

What's $\sum_{n=1}^{\infty} a \cdot r^{n-1}$?

Look at the partial sums:

$$\begin{aligned} S_k &= a + ar + ar^2 + ar^3 + \cdots + ar^{k-1} \\ -rS_k &= \phantom{a +{}} ar + ar^2 + ar^3 + \cdots + ar^{k-1} + ar^k \end{aligned}$$

$$S_k - rS_k = a \qquad\qquad - ar^k$$

$$S_k(1-r) = a(1-r^k)$$

so $\boxed{S_k = a \cdot \frac{1-r^k}{1-r}}$ partial sums of a geometric series (if $r \neq 1$)

To get the whole sum $\sum_{n=1}^{\infty} ar^{n-1}$ we take the limit:

$$\boxed{\lim_{k\to\infty} S_k = \begin{cases} \text{diverges} & \text{if } |r|\geq 1 \\ a\cdot\frac{1}{1-r} & \text{if } |r|<1 \end{cases}}$$ ← Sum of a geometric series

Ex Find the sum of the series $2+\frac{1}{3}+\frac{1}{18}+\frac{1}{108}+\cdots$

$\times\frac{1}{6}$ $\times\frac{1}{6}$ $\times\frac{1}{6}$

This is geometric: $a=2$, $r=\frac{1}{6}$

$$\text{sum} = \frac{a}{1-r} = \frac{2}{1-\frac{1}{6}} = \frac{2}{5/6} = \frac{12}{5}$$

Ex Compute the sum $\displaystyle\sum_{n=1}^{\infty} \frac{3+5^n}{7^n}$.

This is not a geom. series, but can write it as

$$\sum_{n=1}^{\infty} \frac{3}{7^n} + \sum_{n=1}^{\infty} \frac{5^n}{7^n}$$

$$= \sum_{n=1}^{\infty} \frac{3}{7^n} + \sum_{n=1}^{\infty} \left(\frac{5}{7}\right)^n$$

geometric series $a=\frac{3}{7}$, $r=\frac{1}{7}$

geometric series $a=\frac{5}{7}$, $r=\frac{5}{7}$

$$= \frac{3/7}{1-1/7} + \frac{5/7}{1-5/7}$$

$$= \frac{3/7}{6/7} + \frac{5/7}{2/7}$$

$$= \frac{1}{2} + \frac{5}{2} = \underline{3}$$

__Ex__ Does $\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$ converge, and if so, to what?

This is a geometric series: it's $\sum_{n=1}^{\infty} \frac{10 \cdot 10^{n-1}}{(-9)^{n-1}}$

$$= \sum_{n=1}^{\infty} 10 \cdot \left(\frac{10}{-9}\right)^{n-1}$$

geometric, with $a = 10$, $r = -\frac{10}{9}$ $\qquad a_n = ar^{n-1}$

$|r| > 1 \Rightarrow$ __diverges__