Exam 2 average this section 68
overall average (all 408L) 62

Last time: sequences

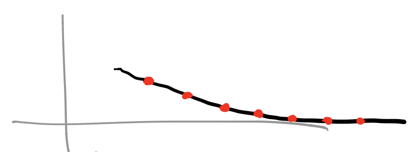
Ex $a_n = \frac{\ln n}{n}$ $0, \frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4}, \frac{\ln 5}{5}, \ldots$

Q 1) Does the seq. $a_n = \frac{\ln n}{n}$ converge? To what?

2) ——— $a_n = \frac{\ln n}{n} - 4$ ——— ?

A 1) The function $f(x) = \frac{\ln x}{x}$

has $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} 1/x = 0.$



So, $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

2) $\lim_{n\to\infty} \frac{\ln n}{n} - 4 = \lim_{n\to\infty} \frac{\ln n}{n} - \lim_{n\to\infty} 4$

$= 0 - 4 = -4$

Q Does the seq. $a_n = \sin\left(\frac{\pi n}{1+4n}\right)$ converge? (to what?)

$$\lim_{n\to\infty} \sin\left(\frac{\pi n}{1+4n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi n}{1+4n}\right)$$
$$= \sin\left(\lim_{n\to\infty} \frac{\pi}{1/n+4}\right)$$
$$= \sin\left(\frac{\pi}{4}\right)$$
$$= \underline{\underline{\frac{\sqrt{2}}{2}}}$$

$$\lim_{n\to\infty} \frac{7x^{80}+4}{9x^{80}-16x^{17}} = \frac{7}{9}$$

Q Does $a_n = 2 + \frac{1}{n} + \frac{n!+1}{(n+1)!}$ converge? (to what?)

Split it up: $\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{n!}{(n+1)!} + \lim_{n\to\infty} \frac{1}{(n+1)!}$

$\frac{n(n-1)(n-2)\cdots(1)}{(n+1)n(n-1)(n-2)\cdots(1)}$

$$= 2 + 0 + \lim_{n\to\infty} \frac{1}{n+1} + 0$$
$$= 2 + 0 + 0 + 0 = \underline{\underline{2}}$$

Ex Does the sequence $a_n = \left(1 - \frac{7}{n}\right)^{-2n}$ converge? (to what?)

If we try to break it up, get $(1)^{-\infty}$

this is an indeterminate form — like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$

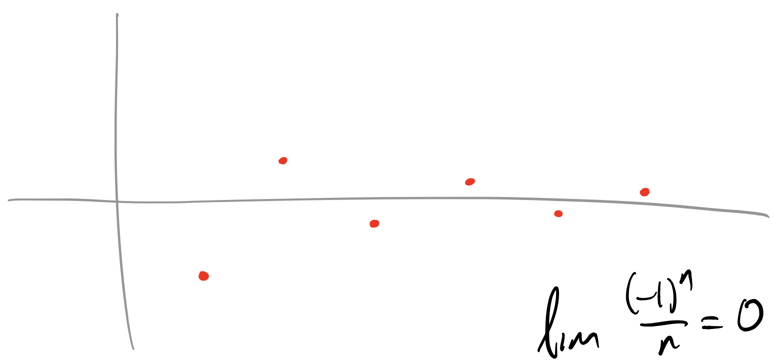
$\longrightarrow$ can't do it by doing the indiv. parts

Use this fact:
$$\boxed{\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x}$$

$$a_n = \left(1-\frac{7}{n}\right)^{-2n} = \left(\left(1-\frac{7}{n}\right)^n\right)^{-2}$$

so $$\lim_{n\to\infty} a_n = \left(\lim_{n\to\infty}\left(1-\frac{7}{n}\right)^n\right)^{-2} = \left(e^{-7}\right)^{-2} = \underline{\underline{e^{14}}}$$

Q $a_n = \frac{(-1)^n}{n}$ does $\{a_n\}$ converge? $a_n = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots$



$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

This is an instance of a rule: if $\lim_{n\to\infty} |a_n| = 0$

then $\lim_{n\to\infty} a_n = 0$.

(here $|a_n| = \frac{1}{n}$, so $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0$

so $\lim_{n\to\infty} a_n = 0$)

NB: $a_n = \frac{(-1)^n}{n} = (-1)^n \cdot \frac{1}{n}$

but: $n^2 = n^5 \cdot \frac{1}{n^3}$

(div. = div. · conv.) — (convergent, divergent, convergent)

So, can't tell whether (convergent)×(divergent) converges or not!

But, it *is* OK to say (convergent)·(convergent) = (convergent)

Ex $\lim_{n\to\infty} \left(\tan^{-1}(n) \cdot \frac{n^2+3}{4n^2+7}\right)$

$= \lim_{n\to\infty} \tan^{-1}(n) \cdot \lim_{n\to\infty} \frac{n^2+3}{4n^2+7}$

$= \frac{\pi}{2} \cdot \frac{1}{4}$

$= \frac{\pi}{8}$

Recall summation notation:

$1+4+9+16+25+36$

$= 1^2+2^2+3^2+4^2+5^2+6^2$

$= \sum_{i=1}^{6} i^2$

(Σ plays a role like $\int$)

Suppose we have a sequence a_n. We try to take the sum of all the terms in the sequence:

Ex $a_n = n$: $\quad 1+2+3+4+5+6+7+\cdots = \sum_{n=1}^{\infty} n$

Ex $a_n = \frac{1}{2^n}$: $\quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$

What do these infinite sums mean?

Like we did for improper $\int$'s: look at the sum of k terms,

$$S_k = \sum_{n=1}^{k} a_n \qquad (\text{"partial sum"})$$

this is another sequence S_k made from the original seq. a_n

Then, we try to take the limit $\lim_{k\to\infty} S_k$.

If that exists we say the sum converges, otherwise it diverges.

Ex if $a_n = n$:

$$\begin{aligned} S_1 &= 1 &&= 1 \\ S_2 &= 1+2 &&= 3 \\ S_3 &= 1+2+3 &&= 6 \\ S_4 &= 1+2+3+4 &&= 10 \\ &\vdots && \vdots \end{aligned}$$

$a_n = 1, 2, 3, 4, \ldots$

$S_k = 1, 3, 6, 10, \ldots$ $\qquad \{S_k\}$ doesn't converge so $\sum_{n=1}^{\infty} n$ doesn't converge

Ex if $a_n = \frac{1}{2^n}$:

$$s_1 = \frac{1}{2} = \frac{1}{2}$$
$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$
$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$
$$\vdots \qquad \vdots$$
$$s_k = \frac{2^k - 1}{2^k}$$

here $\lim_{k\to\infty} s_k = 1$. So, we say $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Rk If $\{a_n\}$ diverges, then $\sum_{n=1}^{\infty} a_n$ always diverges, too.

But if $\{a_n\}$ converges, then $\sum_{n=1}^{\infty} a_n$ might converge, might diverge.

Basic example: geometric sequence

e.g. $a_n = 2, 6, 18, 54, 162, 486, \ldots$

$\times 3 \quad \times 3 \quad \times 3 \quad \times 3 \quad \times 3$

$$a_n = 2 \cdot 3^{n-1}$$

(in general, $a_n = a \cdot r^{n-1} = \left(\frac{a}{r}\right) \cdot r^n$)

a: 1st term

r: ratio between successive terms

What's $\sum_{n=1}^{k} a\cdot r^{n-1}$?

$$S_k = \sum_{n=1}^{k} a\cdot r^{n-1}$$

$$\begin{aligned} S_k &= a + ar + ar^2 + \cdots + ar^{k-1} \\ -\ rS_k &= \phantom{a +{}} ar + ar^2 + \cdots + ar^{k-1} + ar^k \\ \hline S_k - rS_k &= a \phantom{+ ar + ar^2 + \cdots + ar^{k-1}} - ar^k \end{aligned}$$

$$S_k(1-r) = a(1-r^k)$$

so $\boxed{S_k = a\cdot\frac{1-r^k}{1-r}}$ ← partial sum of geometric series

Ex $3+12+48+192 = ?$

geom. seq. $a=3$ first term
$r=4$ common ratio

we want $S_4 = 3\cdot\frac{1-4^4}{1-4} = \frac{3}{-3}\cdot(1-256)$

$= \underline{\underline{255}}$ ✓

To sum the whole (infinite) geometric series:

$$\boxed{\sum_{n=1}^{\infty} a\cdot r^{n-1} = \lim_{k\to\infty} a\cdot\frac{1-r^k}{1-r} = \begin{cases} \text{diverges if } |r|\geq 1 \\ \dfrac{a}{1-r} \text{ if } |r|<1 \end{cases}}$$

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Sum of a geometric series

Q Find the sum of the $\underbrace{\text{the}}_{}$ (infinite) series $2+\frac{1}{3}+\frac{1}{18}+\frac{1}{108}+\cdots$

$\times\frac{1}{6}$

→ this is a geom. series, with $r=\frac{1}{6}$ $\left|\frac{1}{6}\right|<1$ $\Rightarrow \sum$ converges

$a=2$

the sum is $\frac{a}{1-r}=\frac{2}{1-\frac{1}{6}}=\frac{2}{5/6}=\frac{12}{5}$

Q Does the sum $\sum_{n=1}^{\infty}\frac{10^n}{(-9)^{n-1}}$ converge? to what?

this is a geometric series: $a_n=ar^{n-1}$

$$\frac{10^n}{(-9)^{n-1}}=10\cdot\frac{10^{n-1}}{(-9)^{n-1}}$$

$$=10\cdot\left(-\frac{10}{9}\right)^{n-1}$$

geometric, $a=10$

$r=-\frac{10}{9}$

$|r|=\frac{10}{9}>1$ so the sum diverges.

Q Compute $\sum_{n=1}^{\infty} \frac{3+5^n}{7^n}$.

This is not geometric, but it's sum of two:

$$3\sum \frac{1}{7^n} + \sum \frac{5^n}{7^n}$$

$\frac{1}{7^n} = \frac{1}{7}\cdot\left(\frac{1}{7}\right)^{n-1}$

$a = \frac{1}{7}$ $\quad$ $a = 5/7$

$r = \frac{1}{7}$ $\quad$ $r = 5/7$

$$= 3\cdot\frac{\frac{1}{7}}{1-\frac{1}{7}} + \frac{5/7}{1-5/7}$$

$$= 3\cdot\frac{1}{6} + \frac{5}{2} = \underline{\underline{3}}$$