Last time: series $a_1 + a_2 + a_3 + \cdots$

$$= \sum_{n=1}^{\infty} a_n \qquad \left(\text{or } = \sum_{i=1}^{\infty} a_i,\ \text{or } = \sum_{k=1}^{\infty} a_k\right)$$

Could also have sequence beginning from $n=0$ instead of $n=1$ $\quad a_0, a_1, a_2, \ldots$

then the corresponding series would be

$$a_0 + a_1 + a_2 + \cdots$$
$$= \sum_{n=0}^{\infty} a_n$$

Partial sums of $\sum_{n=1}^{\infty} a_n$ are $S_k = \sum_{n=1}^{k} a_n$

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$\vdots$

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k \quad \text{(if it exists)}$$

An example: geometric series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$

$$\left(\text{or: } \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots\right)$$

one way to think of this: a = first term of the series
r = common ratio = ratio of successive terms

and then: partial sum (sum of the first k terms) is $S_k = a \cdot \dfrac{1-r^k}{1-r}$

$$\underset{\sum_{n=0}^{\infty} ar^n \text{ OR } \sum_{n=1}^{\infty} ar^{n-1}}{\overset{\text{sum of the whole series}}{\parallel}} = \begin{cases} \text{converges to } \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges (DNE)} & \text{if } |r| \geq 1 \end{cases}$$

$n=\ \ \ \ \ \ \ \ n=1$

Ex Does $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{3^{2n+2}}$ converge? to what?

this is a geometric series: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^2}\cdot\frac{\pi^n}{3^{2n}}$

$$= \sum \frac{1}{9}\cdot\left(\frac{\pi}{3^2}\right)^n$$

$$= \sum \frac{1}{9}\left(\frac{\pi}{9}\right)^n = \sum_{n=0}^{\infty} ar^n \quad a=1/9 \quad r=\pi/9$$

$|r|<1 \longrightarrow$ converges to $\dfrac{a}{1-r} = \dfrac{1/9}{1-\pi/9} = \underline{\underline{\dfrac{1}{9-\pi}}}$

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from HW: $\displaystyle\sum_{n=0}^{\infty} \frac{(2x-5)^n}{4^n} = \sum_{n=0}^{\infty}\left(\frac{2x-5}{4}\right)^n$ geometric $a=1$, $r=\dfrac{2x-5}{4}$

$\longrightarrow$ converges if $|r|<1$

$$\left|\frac{2x-5}{4}\right|<1$$

$$|2x-5|<4$$

$$|x-\tfrac{5}{2}|<2$$

$$\tfrac{5}{2}-2<x<\tfrac{5}{2}+2$$

$$\tfrac{1}{2}<x<\tfrac{9}{2}$$

if it does converge, it converges to $\dfrac{a}{1-r} = \dfrac{1}{1-\frac{2x-5}{4}}$

4 4

$$= \frac{1}{4-(2x-5)} = \frac{1}{9-2x}$$

Test for Divergence

$a_n = a_1, a_2, a_3, \ldots \qquad \sum_{n=1}^{\infty} a_n$

If $\lim_{n\to\infty} a_n$ doesn't exist, or if it exists but it's not 0,
then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex $\sum_{n=1}^{\infty} \frac{3n+4}{4n-7}$?

$$\lim_{n\to\infty} \frac{3n+4}{4n-7} = \frac{3}{4} \neq 0$$

so this sum diverges.

Ex $1+2+3+4+\cdots$

$= \sum_{n=1}^{\infty} n$

$\lim_{n\to\infty} n$ DNE so this sum diverges.

$\sum_{n=1}^{\infty} n$ diverges.

But, if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ might converge / might diverge

the sequence $\{a_n\}$ converges to 0

Ex $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $1+\frac{1}{4}+\frac{1}{9}+\cdots$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$, so this test gives no info about $\sum_{n=1}^{\infty} \frac{1}{n^2}$

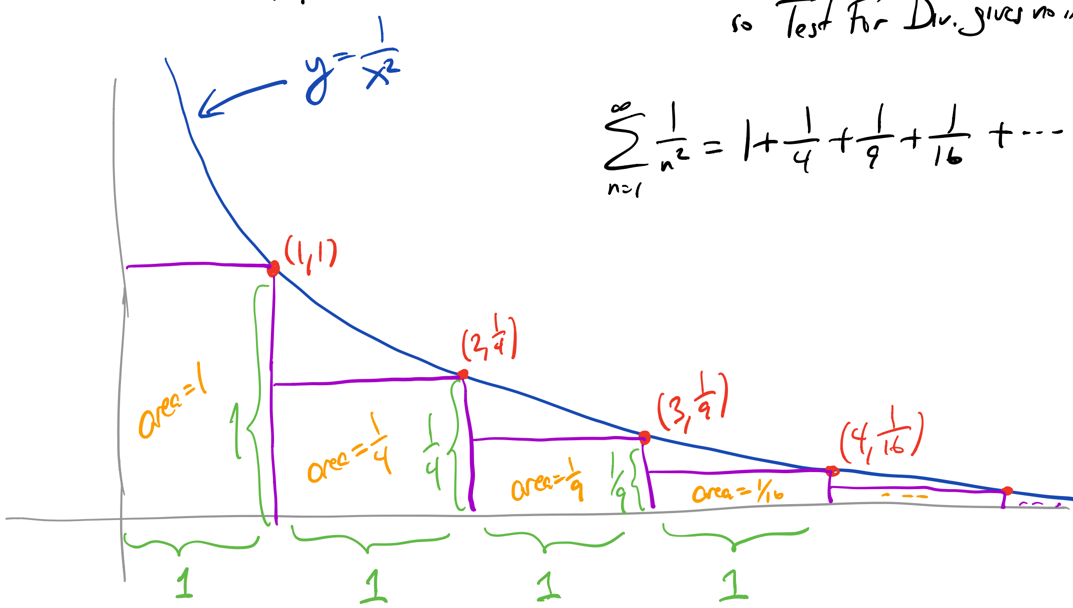
Ex $\quad \sum_{n=1}^{\infty} \frac{1}{n}: \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0,$ so get no info about $\sum_{n=1}^{\infty} \frac{1}{n}$

Integral Test

Look again at $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Not geometric, and $\lim_{n\to\infty} \frac{1}{n^2} = 0$

so Test For Div. gives no info.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$



from this picture: $\left(\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \right) < \underbrace{\int_1^{\infty} dx \, \frac{1}{x^2}}$

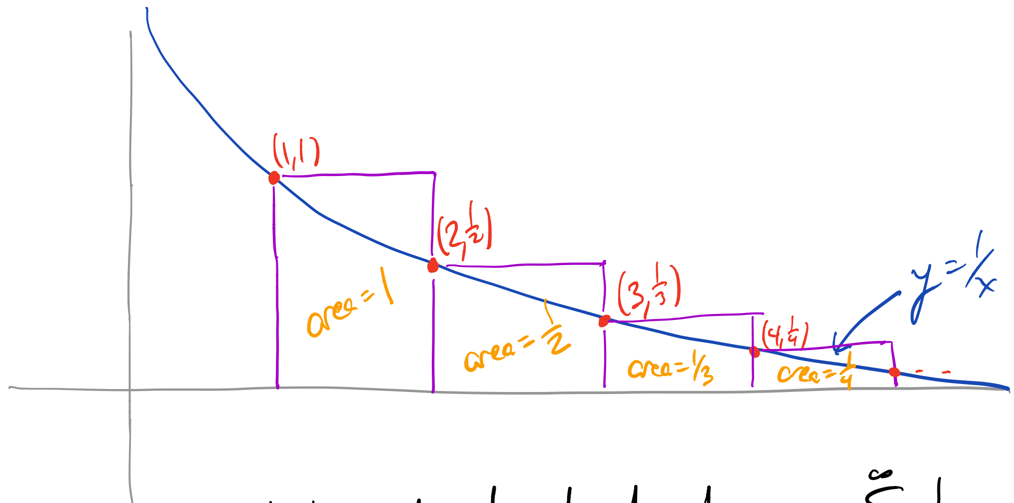
Converges by p-test $(2>1)$ $(\neq 1)$

So, $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ converges

ie $\quad \sum_{n=2}^{\infty} \frac{1}{n^2}$ converges

So $\quad \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$ also converges

How about $\sum_{n=1}^{\infty} \frac{1}{n}$?



total area of rectangles $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$

area under blue curve $= \int_1^{\infty} \frac{1}{x}\, dx$

So, $\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\, dx$

and $\int_1^{\infty} \frac{1}{x}\, dx$ diverges (to ∞) So $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges (to ∞)

General rule (Integral Test):

Suppose $f(x)$ is a continuous, decreasing, positive function defined for $1 \leq x \leq \infty$. Say $a_n = f(n)$.

Then: If $\int_1^{\infty} f(x)\, dx$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\int_1^{\infty} f(x)\, dx$ is divergent then $\sum_{n=1}^{\infty} a_n$ is divergent.

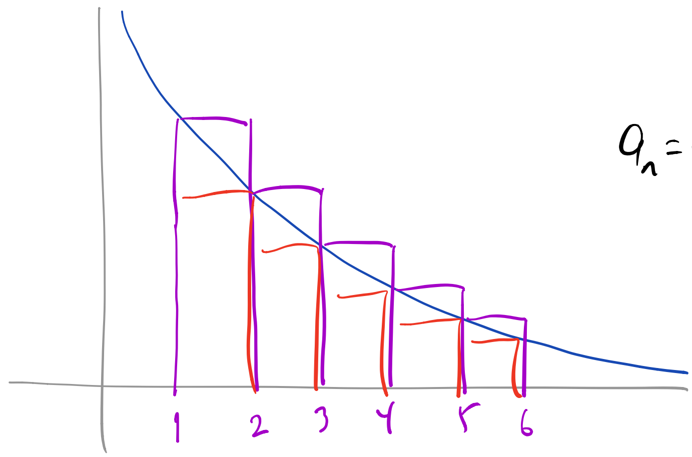
Q. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge?

$\frac{1}{n^2+1} \to 0$ ✓

$\frac{1}{x^2+1}$ is a decreasing positive function, so look at $\displaystyle\int_1^{\infty} \frac{dx}{x^2+1} = \pi/4$

well defined for $x \in [1,\infty)$

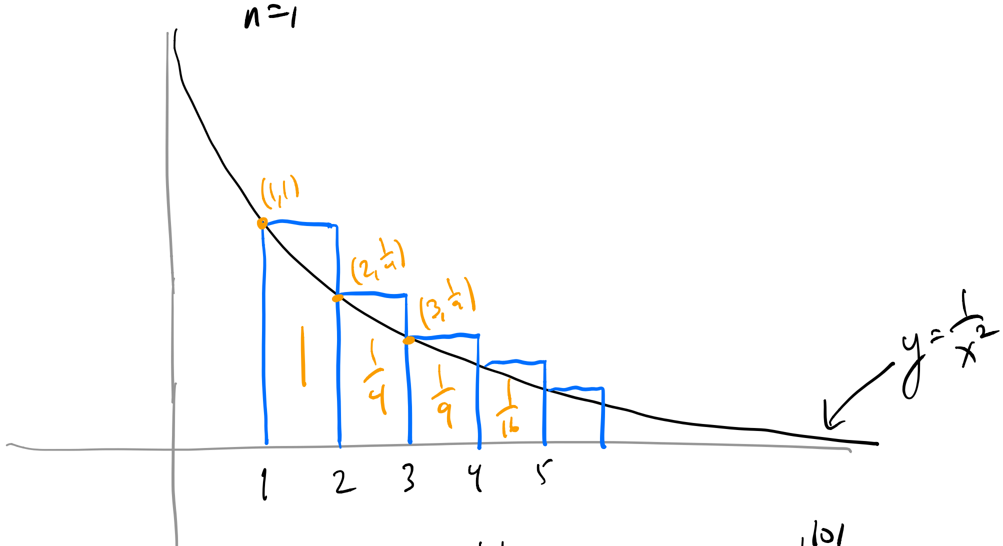
Convergent



$a_n = f(n)$

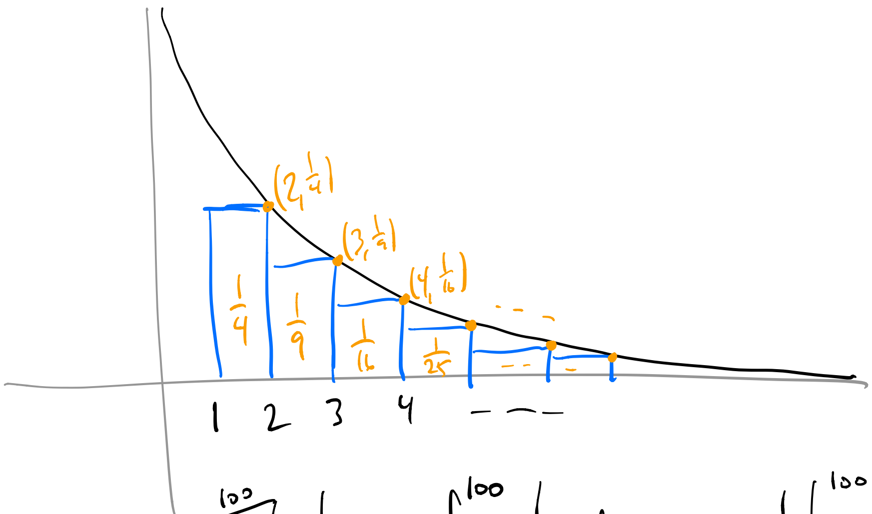
$$\sum_{n=1}^{5} a_n > \int_1^6 f(x)\,dx > \sum_{n=2}^{6} a_n$$

Ex Estimate $\displaystyle\sum_{n=1}^{100} \frac{1}{n^2}$ by comparing to an integral.



$$\sum_{n=1}^{100} \frac{1}{n^2} > \int_1^{101} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{101} = 1 - \frac{1}{101}$$

$$= \frac{100}{101}$$



$$\sum_{n=2}^{100} \frac{1}{n^2} < \int_1^{100} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{100}$$

$$= 1 - \frac{1}{100} = \frac{99}{100}$$

$$\sum_{n=2}^{100} \frac{1}{n^2} < \frac{99}{100}$$

$$\sum_{n=1}^{100} \frac{1}{n^2} < \frac{199}{100}$$

So, $\frac{100}{101} < \sum_{n=1}^{100} \frac{1}{n^2} < \frac{199}{100}$