Last time: if we have a sequence $a_1, a_2, a_3, \ldots$

then we can make the series $a_1 + a_2 + a_3 + \cdots$

$$= \sum_{n=1}^{\infty} a_n \quad \left(= \sum_{i=1}^{\infty} a_i, \ = \sum_{k=1}^{\infty} a_k, \text{ etc.} \right)$$

[Could also have sequences beginning from $n=0$ instead of $n=1$,

$$a_0 + a_1 + a_2 + \cdots = \sum_{n=0}^{\infty} a_n$$
]

Partial sums of $\sum_{n=1}^{\infty} a_n$ are $S_k = \sum_{n=1}^{k} a_n$

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$

and $\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k$ (if it exists)

One class of examples: geometric series $\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + ar + ar^2 + \cdots$

a = first term of series

r = common ratio

$$\sum_{n=0}^{\infty} a \cdot r^n = a + ar + ar^2 + \cdots$$

Partial sums (sum of first k terms)

$$S_k = \sum_{n=1}^{k} a \cdot r^{n-1} = a \cdot \frac{1-r^k}{1-r}$$

Total sum

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \begin{cases} \text{converges to } \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Q Does $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{2n+2}}$ converge? To what?

— this is a geom. series: n only appears as an exponent (something)^n

$$\sum_{n=0}^{\infty} \frac{\pi^n}{3^{2n+2}} = \sum_{n=0}^{\infty} \frac{1}{3^2} \cdot \frac{\pi^n}{3^{2n}}$$ $a \cdot r^n$

$$= \sum_{n=0}^{\infty} \frac{1}{9} \cdot \left(\frac{\pi}{3^2}\right)^n$$ geometric: $a = \frac{1}{9}$, $r = \frac{\pi}{9}$

converges, because $\left|\frac{\pi}{9}\right| < 1$

converges to $\frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{\pi}{9}} = \frac{1}{9-\pi}$

Q Does $\sum_{n=0}^{\infty} \frac{(2x-5)^n}{4^n}$ converge? To what?

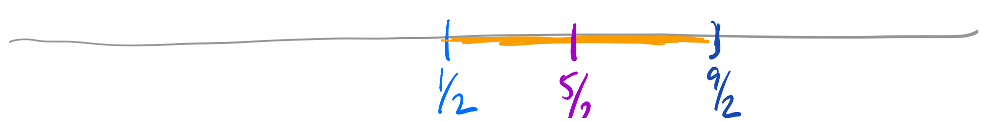
This is geometric: $\sum_{n=0}^{\infty} \left(\frac{2x-5}{4}\right)^n$ $a = 1$, $r = \frac{2x-5}{4}$

$\Longrightarrow$ converges if $\left|\frac{2x-5}{4}\right| < 1$

$$|2x-5| < 4$$

$$\left|x - \tfrac{5}{2}\right| < 2$$

$$\tfrac{1}{2} < x < \tfrac{9}{2}$$



<u>if</u> it converges, it converges to $a \cdot \frac{1}{1-r} = \frac{1}{1-\frac{2x-5}{4}}$

$$= \frac{4}{4-(2x-5)} = \frac{4}{9-2x}$$

$$\lim_{n\to\infty} n\left(f\left(\tfrac{n+1}{n}\right) - f(1)\right) \qquad f(x) = 5x^2+3x+4$$

$$= \lim_{n\to\infty} n\cdot\left(5\left(\tfrac{n+1}{n}\right)^2 + 3\left(\tfrac{n+1}{n}\right) + 4 - 12\right)$$

$$= \lim_{n\to\infty} n\cdot\left(\frac{5(n+1)^2}{n^2} + \frac{3n(n+1)}{n^2} + \frac{-8n^2}{n^2}\right)$$

$$= \lim_{n\to\infty} n\cdot\frac{5n^2+10n+5+3n^2+3n-8n^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{13n+5}{n} = \underline{\underline{13}}$$

<u>OR</u>: $\lim_{n\to\infty} n\left(f\left(\tfrac{n+1}{n}\right) - f(1)\right) = \lim_{n\to\infty} \frac{f(1+\frac{1}{n}) - f(1)}{\left(\frac{1}{n}\right)} = \underset{\overset{\shortparallel}{13}}{f'(1)}$

recall $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Q $\displaystyle\sum_{n=1}^{\infty} \frac{3n+4}{4n-7}$ — converges or not?

Note $\displaystyle\lim_{n\to\infty} \frac{3n+4}{4n-7} = \frac{3}{4}$. So, by Test For Divergence, $\displaystyle\sum_{n=1}^{\infty} \frac{3n+4}{4n-7}$ diverges.

Test For Divergence:
If the sequence $\{a_n\}$ converges to something other than 0 or diverges
then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Ex $\displaystyle\sum_{n=1}^{\infty} n$ diverges

$\displaystyle\sum_{n=1}^{\infty} (-1)^n$ diverges

$\displaystyle\sum_{n=1}^{\infty} (-1)^n n$ diverges

all by TFD.

Warning: if the seq. $\{a_n\}$ does converge to 0, then TFD doesn't tell us anything about whether $\displaystyle\sum_{n=1}^{\infty} a_n$ converges or diverges!

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$

so TFD gives no info

Ex $\sum_{n=1}^{\infty} \frac{1}{n}$: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ $\lim_{n\to\infty} \frac{1}{n} = 0$

so TFD gives <u>no info</u>

Remark: these are <u>not</u> geometric series! Geom. series are $\sum ar^n$

How to tell if they converge or not?

<u>Integral Test</u>

Suppose $f(x)$ is a function which is <u>positive</u>, <u>decreasing</u>, <u>continuous</u> for $1 \le x < \infty$. Say $a_n = f(n)$.

Then, $\sum_{n=1}^{\infty} a_n$ <u>converges</u> if $\int_1^{\infty} f(x)\,dx$ <u>converges</u>

$\sum_{n=1}^{\infty} a_n$ <u>diverges</u> if $\int_1^{\infty} f(x)\,dx$ <u>diverges</u>

Q 1) Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge? 2) Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

3) Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge?

A 1) $\sum_{n=1}^{\infty} \frac{1}{n}$: $f(x) = \frac{1}{x}$ is <u>positive</u>, <u>continuous</u>, <u>decreasing</u> for $1 \le x \le \infty$

so can use Int. Test.

$\int_1^{\infty} \frac{1}{x}\,dx$ <u>diverges</u>

by p-test, so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

2) $\sum \frac{1}{n^2}$: same idea $\int_1^\infty \frac{1}{x^2}\,dx$ converges

so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

3) $\sum \frac{1}{n^2+1}$: same idea, $\int_1^\infty \frac{1}{x^2+1} = \lim_{t\to\infty} \tan^{-1} x \Big|_1^t$

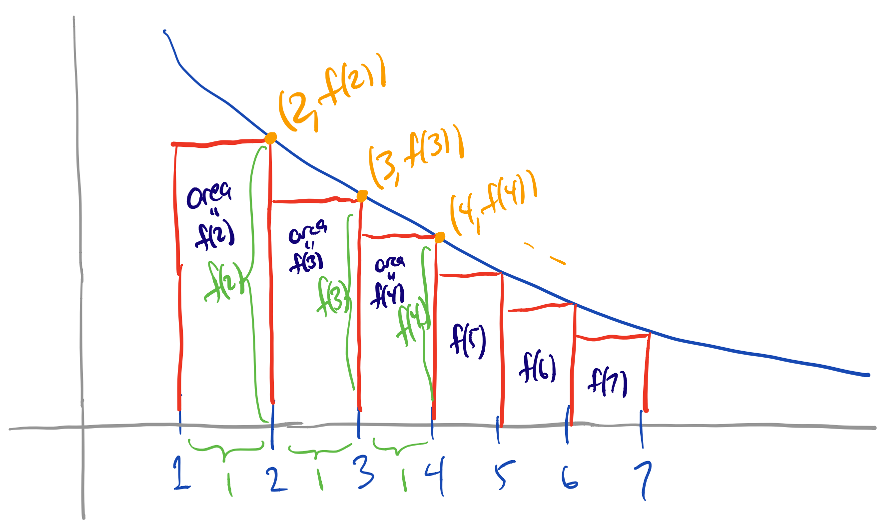
$= \lim_{t\to\infty} \tan^{-1} t - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$

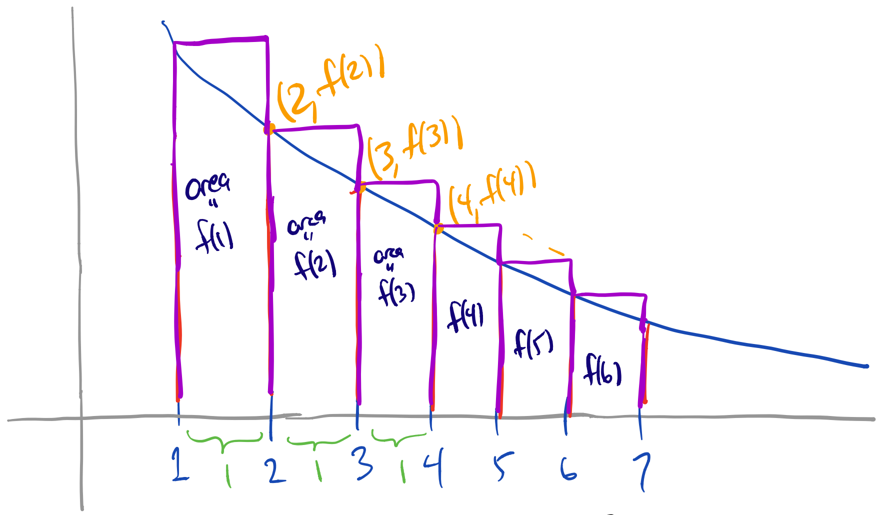
So $\sum \frac{1}{n^2+1}$ converges

Remark: once we know 2) converges, could show 3) converges using Comparison Test (next lecture) since $\frac{1}{n^2+1} < \frac{1}{n^2}$

Using integrals to estimate sums



So if $a_n = f(n)$ then $\sum_{n=2}^{7} a_n < \int_1^7 f(x)\, dx$



so $\sum_{n=1}^{6} a_n > \int_1^7 f(x)\, dx$