From HW: Does $\sum_{n=0}^{\infty} \frac{3}{\ln(n+5)} \cos\left(\frac{n\pi}{2}\right)$ converge?

$\cos\left(\frac{n\pi}{2}\right) = 1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots$

so

$\frac{3}{\ln(n+5)} \cos\left(\frac{n\pi}{2}\right) = \frac{3}{\ln 5}, 0, -\frac{3}{\ln 7}, 0, \frac{3}{\ln 9}, 0, -\frac{3}{\ln 11}, \ldots$

so the sum is $\frac{3}{\ln 5} - \frac{3}{\ln 7} + \frac{3}{\ln 9} - \frac{3}{\ln 11} + \cdots$

alternating series. The terms $\frac{3}{\ln 5}, \frac{3}{\ln 7}, \frac{3}{\ln 9}, \ldots$ are decreasing, have limit $= 0$

So $\sum_{n=0}^{\infty} \frac{3}{\ln(n+5)} \cos\left(\frac{n\pi}{2}\right)$ converges.

Absolute Convergence

$\sum_{n=1}^{\infty} a_n$ Call $\sum_{n=1}^{\infty} a_n$ "absolutely convergent" if $\sum_{n=1}^{\infty} |a_n|$ converges.

Q 1) Is $\sum_{n\geq 1}^{\infty} \frac{(-1)^n}{n^2+1}$ absolutely convergent? Convergent?

2) Is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ absolutely convergent? Convergent?

A) 1) Absolute convergence: $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$, convergent (lim-comp. to $\frac{1}{n^2}$ p-test, $2>1$)

So $\sum \frac{(-1)^n}{n^2+1}$ is absolutely convergent

Convergence: $\sum \frac{(-1)^n}{n^2+1}$ is alternating series and $\frac{1}{n^2+1}$ is decreasing, has limit $= 0$

$\Rightarrow$ convergent by alt. series test

(Or: just use absolutely conv $\Rightarrow$ convergent)

2) Absolute convergence: $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, divergent (p-test, $\frac{1}{2} < 1$)

So $\sum \frac{(-1)^n}{\sqrt{n}}$ is <u>not absolutely convergent</u>

Convergence: $\sum \frac{(-1)^n}{\sqrt{n}}$ is <u>alternating series</u> and $\frac{1}{\sqrt{n}}$ is <u>decreasing</u>, has <u>limit = 0</u>

$\Longrightarrow$ <u>convergent</u> by alt. series test

Fact: If $\sum_1^{\infty} a_n$ is absolutely convergent, then $\sum_1^{\infty} a_n$ is convergent.

If $\sum_1^{\infty} a_n$ is convergent but <u>not</u> absolutely convergent, then we call it <u>conditionally convergent</u>.

So, 3 possibilities:

- absolutely convergent
- conditionally convergent
- divergent

<u>Q</u> Is $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ <u>abs. conv</u>, <u>cond. conv.</u>, or <u>divergent</u>?

Absolute convergence: $\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ $\qquad |\cos(n)| \leq 1$

<u>convergent</u> (p-test)

<u>So</u>, $\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$ is <u>convergent</u> by Comparison Test.

So, $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is <u>absolutely convergent</u>.

(In particular, $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is <u>convergent</u>.)

Q $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)+7}$. abs convergent, cond convergent, or divergent?

Absolutely convergent? $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)+7}$

one way: Integral Test (but hard!)

another way: $\ln(n) < n$

e.g. if $n = 1000$, $\ln(n) \approx 7$

even $\ln(n) < \sqrt{n}$

$\ln(n) < n^{1/10}$

$$\ln\ln n < \ln n < \cdots < n^{1/10} < \sqrt{n} < n < n^2 < n^3 < \cdots < e^n < \cdots < e^{n^2} < \cdots$$

So: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)+7} > \underbrace{\sum_{n=2}^{\infty} \frac{1}{n}}_{\text{divergent}}$

so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)+7}$ diverges by Comp. Test

So, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n+7}$ is not absolutely convergent

Is it cond. convergent? It's an alternating series so look at $\dfrac{1}{\ln n+7}$ —

that's decreasing, limit $=0$

So $\displaystyle\sum \frac{(-1)^n}{\ln n+7}$ is convergent by Alt. Series Test

So, $\displaystyle\sum \frac{(-1)^n}{\ln n+7}$ is cond. convergent

Ratio Test

1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \underset{(\text{or } \infty)}{L} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

(If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the test gives no info.)

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$. Ratio test: $a_n = (-1)^n \frac{n^3}{3^n}$

$a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{3^n}{3^n \cdot 3}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3 \cdot \frac{1}{3}$$

$$= 1 \cdot \frac{1}{3} = \frac{1}{3} < 1$$

So $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ converges (even absolutely conv.)

Q Does $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converge? $a_n = \frac{n^n}{n!}$ $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}$$

$$= \frac{(n+1)^{n+1}}{n^n} \cdot \underbrace{\frac{n!}{(n+1)!}}_{\to \frac{n(n-1)(n-2)\cdots 1}{(n+1)n(n-1)(n-2)\cdots 1}}$$

$$= (n+1)\cdot\frac{(n+1)^n}{n^n}\cdot\frac{1}{n+1}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

and $\underline{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1}$

So $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$ is <u>divergent</u> !