Last time: <u>Ratio Test</u> for $\sum_{n=1} a_n$

Look at $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

if $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

if $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

if $L = 1$ then Ratio Test gives no info

Q. • What happens if we apply Ratio Test to $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$?

• Does $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$ converge or diverge?

A. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{n+1}/1+(n+1)^2}{\sqrt{n}/1+n^2} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2}$

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot \frac{1+n^2}{1+n^2+2n+1}$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{n^2+1}{n^2+2n+2}$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}$$

$$= 1 \cdot 1$$

$$= \underline{\underline{1}}$$ so Ratio Test <u>gives no info</u>

[Remark similar for $a_n = \frac{n^4+3n+1}{n^6+7n+8}$, $a_n = \frac{P(n)}{Q(n)}$ P, Q polynomials]

So, need to use another test.

$$\sum \frac{\sqrt{n}}{1+n^2} < \sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$$

So, $\sum \frac{\sqrt{n}}{1+n^2}$ converges by Comparison Test

convergent by p-test $\left(\frac{3}{2}>1\right)$

(also conv. absolutely)

Remark $\sum \frac{\sqrt{n}}{n^2-1}$ also converges, by Limit-Comp

$$a_n = \frac{\sqrt{n}}{n^2-1} \qquad b_n = \frac{\sqrt{n}}{n^2}$$

$$\left(\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\sqrt{n}/n^2-1}{\sqrt{n}/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1\right)$$

Root Test $\sum_{n=1}^{\infty} a_n$

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ (or $=\infty$) then $\sum_{n=1}^{\infty} a_n$ is divergent

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then the test gives no info.

Q. 1) Is $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ convergent?

2) Is $\sum_{n=1}^{\infty} \left(\frac{3n^2+4}{n^3}\right)^{5n}$ conv.?

1) root test: $\sqrt[n]{\left|\frac{2n+3}{3n+2}\right|^n} = \frac{2n+3}{3n+2}$ $\lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$

so $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ <u>converges</u>

2) root test: $\sqrt[n]{\left(\frac{3n^2+4}{n^3}\right)^{5n}} = \left(\frac{3n^2+4}{n^3}\right)^5 \sim \frac{n^{10}}{n^{15}}$ so $\lim_{n\to\infty} \left(\frac{3n^2+4}{n^3}\right)^5 = 0$

or: $\sim \left(\lim_{n\to\infty} \frac{3n^2+4}{n^3}\right)^5 = 0^5 = 0$

$0 < 1$ so $\sum \left(\frac{3n^2+4}{n^3}\right)^{5n}$ <u>converges</u>

<u>Strategy For Testing Series</u> $\sum a_n$

Classify the series according to its <u>form</u>.

1) $\sum \frac{1}{n^p}$: p-test. $\begin{cases} \text{conv. if } p>1 \\ \text{div. if } p \le 1 \end{cases}$

2) $\sum ar^{n-1}$ or $\sum ar^n$: geometric $\begin{cases} \text{conv. if } |r|<1 \\ \text{div. if } |r| \ge 1 \end{cases}$

3) If the series <u>looks similar</u> to p-series or geometric series:
try <u>comparison</u> or <u>limit-comparison</u> (taking b_n to be the p-series or geometric series)
(If the series has some negative terms, apply this method instead to $\sum |a_n|$ — i.e. test for <u>absolute</u> convergence)

4) If you can see easily that $\lim_{n\to\infty} a_n \neq 0$, use Test For Divergence

5) If series is of form
$\sum (-1)^n b_n$ or $\sum (-1)^{n+1} b_n$ try Alternating Series Test.

6) If the series involves factorials (or other products with n terms like k^n), try Ratio Test.

[But not for series where a_n is just a rational function like $a_n = \frac{5n^2+6n}{3n^2+4}$, Ratio Test gives no info for these]

7) If $a_n = (\text{something})^n$ use Root Test

8) If $a_n = f(n)$ and you know how to do $\int_1^\infty f(x)\,dx$

and $f(x)$ is decreasing for large x

try Integral Test.

Q. Do these converge?

- $\sum_{n=1}^{\infty} \frac{n+8}{2n+1}$
- $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
- $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

TFD: $\lim_{n\to\infty} \frac{n+8}{2n+1} = \frac{1}{2} \neq 0$, so Σ divergent

or lim-comp to $\sum_{n=1}^{\infty} \frac{1}{2}$ which is divergent

ratio test: $\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!}$

$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} 2 \cdot \frac{1}{n+1}$

$= \underline{0} < 1$

so, Σ converges

integral test:

$f(x) = x^2 e^{-x^3}$

$\int_1^\infty f(x)\,dx$

$= \int_1^\infty x^2 e^{-x^3}\,dx$

$u = x^3$

$du = 3x^2\,dx \quad \frac{du}{3} = x^2\,dx$

$\to \int_1^\infty e^{-u}\,\frac{du}{3}$

$= -\frac{1}{3} e^{-u} \Big|_1^\infty$

$$= \lim_{t\to\infty} -\frac{1}{3}e^{-u}\Big|_1^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{3}e^{-t}+\frac{1}{3}e^{-1}\right)$$

$$= 0+\frac{1}{3}e^{-1}$$

convergent

so $\sum$ converges

Q $\cdot \sum \frac{\sin^2 n}{3^n}$ $\quad \cdot \sum n\sin\left(\frac{1}{n}\right)$ $\quad \cdot \sum (-1)^n \frac{n^3}{n^4+1}$

convergent or divergent?

$0 \le \frac{\sin^2 n}{3^n} < \frac{1}{3^n}$

and $\sum \frac{1}{3^n}$ converges (Geometric, $r=\frac{1}{3}<1$)

so $\sum \frac{\sin^2 n}{3^n}$ conv. by Comparison Test

$\lim_{n\to\infty} \underset{\infty}{n} \, \underset{\sin(0)=0}{\sin\left(\frac{1}{n}\right)}$

$= \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{-\frac{1}{n^2}\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}}$

$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$

So, by TFD, $\sum$ diverges

(principle: for small x, $\sin(x) \approx x$)

convergent by Alt. Series Test,

$b_n = \frac{n^3}{n^4+1}$ has $\lim_{n\to\infty} b_n = 0$ and b_n decreasing

(but: not absolutely conv. by Lim-Comp:

$|a_n| = \frac{n^3}{n^4+1}$

$b_n = \frac{n^3}{n^4} = \frac{1}{n}$

and $\sum b_n$ diverges by p-test)