Last time: Ratio Test $\sum_{n=1}^{\infty} a_n$

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

(If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then Ratio Test gives no info)

Q Do the series $\sum \frac{\sqrt{n}}{1+n^2}$ and $\sum n \cdot \frac{1}{4^n}$ converge?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n+1}/1+(n+1)^2}{\sqrt{n}/1+n^2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2}$$

morally, $\frac{\sqrt{n}}{\sqrt{n}} \cdot \frac{n^2}{n^2} = 1$ for large n

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{n^2+1}{(n+1)^2+1}$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{1+\frac{1}{n^2}}{(1+\frac{1}{n})^2+\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} 1 \cdot \frac{1}{1} = 1$$

$\to$ ratio test gives no info

use Lim-Comp:

$$a_n = \frac{\sqrt{n}}{1+n^2} \qquad b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

and $\sum_{n=1}^{\infty} b_n$ converges by p-test

so $\sum a_n$ also converges (abs)

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)/4^{n+1}}{n/4^n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{4^n}{4^{n+1}} \leftarrow \frac{4^n}{4^n \cdot 4}$$

$$= 1 \cdot \frac{1}{4} = \frac{1}{4} < 1$$

$\Rightarrow \sum$ converges (abs) by Ratio Test

Root Test

$$\sum a_n$$

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ then $\sum a_n$ diverges.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $\sum a_n$ converges absolutely.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then the Root Test gives no info.

Q 1) Does $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ converge?

2) Does $\sum_{n=1}^{\infty} \left(\frac{3n^2}{4n^3+1}\right)^{5n}$ converge?

1) root test: $\sqrt[n]{\left|\frac{2n+3}{3n+2}\right|^n} = \frac{2n+3}{3n+2}$ $\quad \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$

So $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)$ converges

2) root test: $\sqrt[n]{\left(\frac{3n^2}{4n^3+1}\right)^{5n}} = \left(\frac{3n^2}{4n^3+1}\right)^{5n\cdot\frac{1}{n}} = \left(\frac{3n^2}{4n^3+1}\right)^5$

$$\lim_{n\to\infty}\left(\frac{3n^2}{4n^3+1}\right)^5 = \left(\lim_{n\to\infty}\frac{3n^2}{4n^3+1}\right)^5$$

$$= 0^5 = 0$$

So $\sum \left(\frac{3n^2}{4n^3+1}\right)^{5n}$ converges

Strategy For Testing Series

$\sum a_n$

Classify the series according to its form.

1) $\sum_1 \frac{1}{n^p}$: p-test. $\begin{cases} \text{conv if } p>1 \\ \text{div if } p \leq 1 \end{cases}$

2) $\sum_1 ar^{n-1}$ or $\sum_1 ar^n$: geometric $\begin{cases} \text{conv. if } |r|<1 \\ \text{div. if } |r| \geq 1 \end{cases}$

3) If the series looks similar to $\frac{1}{n^p}$ or geometric:
   try comparison or lim-comparison with
   $b_n = \frac{1}{n^p}$ or geometric.

   (If the series has some negative terms then apply this method instead to $\sum |a_n|$ — ie test for absolute convergence.)

4) If you can see that $\lim\limits_{n\to\infty} a_n \neq 0$, use TFD.

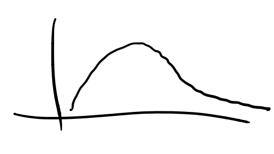
5) If the series is of form $\sum (-1)^n b_n$ or $\sum (-1)^{n+1} b_n$
   try Alternating Series Test.

6) If the series involves factorials, or other products with n terms, like k^n try Ratio Test.

   [But not for series where a_n is just a rational function like $a_n = \frac{5n^2+7}{8n^6+4}$ — Ratio Test will be useless for these]

7) If $a_n = (\text{something})^n$ try Root Test

8) If $a_n = f(n)$ and you know how to do $\int_1^\infty f(x)\,dx$
   and $f(x)$ is positive, decreasing (for large x)
   try Integral Test.



Q Which converge?

- $\sum \frac{n+8}{2n+1}$
- $\sum \frac{2^n}{n!}$
- $\sum n^2 e^{-n^3}$

$\lim_{n\to\infty} \frac{n+8}{2n+1} = \frac{1}{2}$

so the $\sum$ diverges by TFD

ratio test:

$$\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!}$$

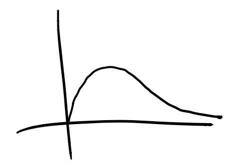
$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{1}{n+1}$$

$$= 0 < 1$$

$\rightarrow \sum$ convergent

$\int$ test: $f(x) = x^2 e^{-x^3}$



$$\int_1^\infty dx\; x^2 e^{-x^3}$$

$u = x^3$

$du = 3x^2 dx$

$\frac{du}{3} = x^2 dx$

$$= \int_1^\infty \frac{du}{3} e^{-u}$$

$$= -\frac{1}{3} e^{-u} \Big|_1^\infty$$

$$= \lim_{t\to\infty} -\frac{1}{3} e^{-u} \Big|_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{3} e^{-t} + \frac{1}{3} e^{-1}$$

$$= 0 + \frac{1}{3} e^{-1}$$

$$= \frac{1}{3e}$$

because the $\int$ converges, the $\sum$ converges ✓

Remark: $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

$$e^{-n^3} < \frac{1}{n^4}$$

$e^{n^3} > n^4$

use comparison test:

$$n^2 e^{-n^3} < n^2 \cdot \frac{1}{n^4} = \frac{1}{n^2}$$

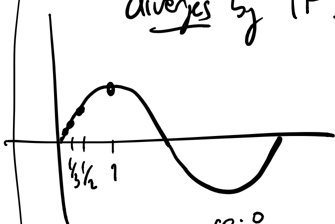
$\sum \frac{1}{n^2}$ converges

so $\sum n^2 e^{-n^3}$ converges.

Ratio Test: $\lim\limits_{n\to\infty} \dfrac{(n+1)^2 e^{-(n+1)^3}}{n^2 e^{-n^3}} = \cdots = 0$

Q Which converges?

- $\sum n \sin\left(\frac{1}{n}\right)$ — diverges by TFD!
- $\sum \dfrac{1}{2+3^n}$ — converges by lim-comp to $\frac{1}{3^n}$
- $\sum (-1)^n \dfrac{n^3}{n^4+1}$ — converges cond by Alt Seri Test and lim-comp to $\frac{1}{n}$



$\sin(x) \approx x$
x small

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right)$$

$$= 1$$