Power Series

A power series centered at a is a series of the form

$$\sum_{n=0}^{\infty} \underset{\text{"constant"}}{\underset{\uparrow}{c_n}}\,(\underset{\text{variable}}{\underset{\uparrow}{x}}-\underset{\text{"constant"}}{\underset{\nwarrow}{a}})^n$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

is a power series centered at $a=0$

with $c_n = \frac{1}{n}$

why is this important?

it converges to

$-\ln(1-x)$

if $|x|<1$

e.g. $-\ln(0.99)$

$= (0.01) + \frac{(0.01)^2}{2} + \cdots$

$\approx 0.01005\ldots$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{2n} = \frac{x-1}{2} + \frac{(x-1)^2}{4} + \frac{(x-1)^3}{6} + \cdots$

is a power series centered at $a=1$

with $c_n = \frac{1}{2n}$

Ex $\displaystyle\sum_{n=0}^{\infty} n!\,(x-4)^n = 1 + (x-4) + 2(x-4)^2 + 6(x-4)^3 + \cdots$

$[0! = 1]$

is a power series centered at $a=4$

with $c_n = n!$

Fact For a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, 3 possibilities:

1) Series converges only for $x=a$.

2) Series converges for every x.

3) There is some number $R>0$ such that

the series converges for $|x-a|<R$

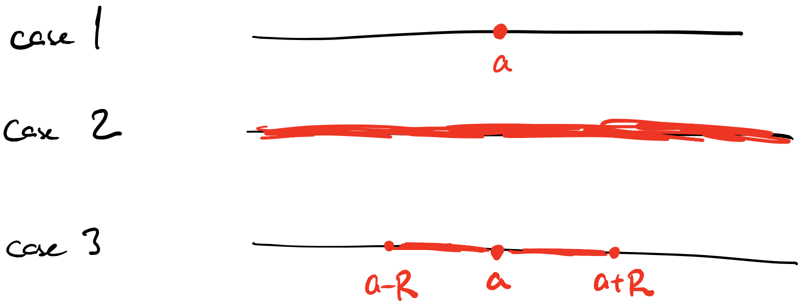
diverges for $|x-a|>R$

We call R the "radius of convergence"

In case 1, we say $R=0$.

In case 2, we say $R=\infty$.

The "interval of convergence" is the set of all x where the series converges.



In case 3, 4 possibilities for the interval:

$(a-R,\ a+R)$ $\quad$ $(a-R,\ a+R]$

$[a-R,\ a+R)$ $\quad$ $[a-R,\ a+R]$

Q What is the interval of convergence for $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$?

Ratio Test:

if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, Σ converges

if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, Σ diverges

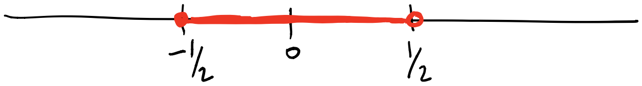
Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x)^{n+1}/n+1}{(2x)^n/n}\right| = \left|\frac{(2x)^{n+1}}{(2x)^n}\right| \cdot \left|\frac{n}{n+1}\right|$

$= |2x| \cdot \frac{n}{n+1}$

$$\lim_{n\to\infty} |2x| \cdot \frac{n}{n+1} = |2x|$$

So, Ratio Test says: Σ converges if $|2x|<1$ i.e. $|x|<\frac{1}{2}$

diverges if $|2x|>1$ i.e. $|x|>\frac{1}{2}$



Still need to know what happens if $|x| = 1/2$

ie $x = 1/2$ or $x = -1/2$

If $x = 1/2$: $\displaystyle\sum_{n=1}^{\infty} \frac{(2 \cdot 1/2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$x = -1/2$: $\displaystyle\sum_{n=1}^{\infty} \frac{(2 \cdot (-1/2))^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

So, the interval is $[-1/2, \frac{1}{2})$

Q What is the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$?

Ratio Test: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\ |x|^{n+1}}{n!\ |x|^n} = \frac{(n+1)!}{n!} \quad \frac{|x|^{n+1}}{|x|^n}$

$= (n+1)\,|x|$

$\displaystyle\lim_{n\to\infty} (n+1)|x| = \begin{cases} \infty & \text{if } |x| \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

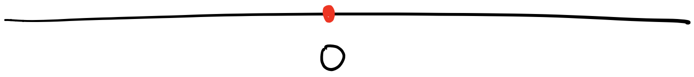
$\displaystyle\lim_{n\to\infty} (n+1) \cdot 0$

$\|$

$\displaystyle\lim_{n\to\infty} 0 = 0$

So, Ratio Test $\Rightarrow$ Σ converges if $x = 0$

Σ diverges if $x \neq 0$



interval of convergence is one point $\{0\}$.

radius of convergence is $R = 0$.

Q $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ interval of convergence?

Ratio test: $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+2}/2^{2n+2}((n+1)!)^2}{|x|^{2n}/2^{2n}(n!)^2}$$

$$= \frac{|x|^{2n+2}}{|x|^{2n}} \cdot \frac{2^{2n}}{2^{2n+2}} \cdot \frac{(n!)^2}{((n+1)!)^2}$$

$$= |x|^2 \cdot \frac{1}{2^2} \cdot \frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty} \frac{|x|^2}{4} \cdot \frac{1}{(n+1)^2} = 0 < 1$$

So ratio test $\Rightarrow$ $\displaystyle\sum \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ converges for <u>all</u> x ("Bessel function")

Interval of convergence: $(-\infty, \infty)$

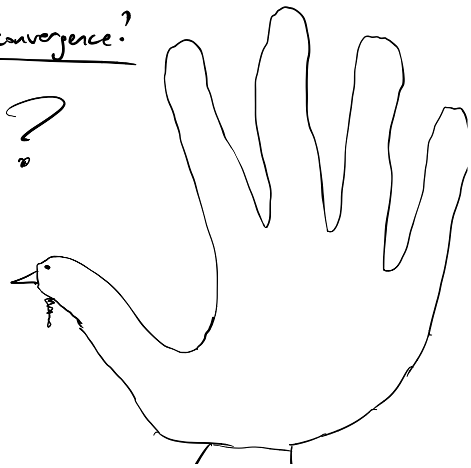
radius of convergence: $R = \infty$

Q $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ what is <u>interval of convergence</u>?

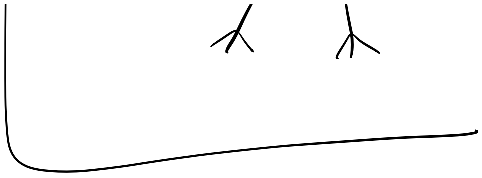
Ratio test: $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\,|x+2|^{n+1}/3^{n+2}}{(n)\ |x+2|^n/3^{n+1}}$$

$$= \frac{n+1}{n} \cdot |x+2| \cdot \frac{1}{3}$$

$$\lim_{n\to\infty} \frac{n+1}{n} \cdot |x+2| \cdot \frac{1}{3} = \frac{1}{3}|x+2|$$



$\Rightarrow$ $\sum$ converges <u>if</u> $\frac{1}{3}|x+2|<1$



i.e. $|x+2|<3$

"distance between x and -2"

radius of convergence $= \underline{\underline{3}}$

check endpoints: $x=1 \longrightarrow$ diverges by TFD

$x=-5 \longrightarrow$ diverges by TFD

interval of cnv. is

$(-5,1)$