HW13 due Dec 5, also Exam 3 that day
next LM due midnight Nov 27

Power Series

A "power series centered at a" is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

"constant" variable "const"

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

is a power series centered at $a=0$
with $c_n = \frac{1}{n}$.

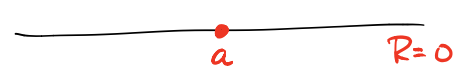
this series converges if
$-1 \leq x < 1$
it converges to $-\ln(1-x)$.
ex $-\ln(0.99)$: plug in $x = 0.01$
then
$-\ln(0.99) = 0.01 + \frac{(0.01)^2}{2} + \cdots$
$\ln(0.99) = -0.01005\ldots$

Ex $\sum n!\,(x-4)^n = 1 + (x-4) + 2(x-4)^2 + 6(x-4)^3 + \cdots$

is a power series centered at $a=4$
$c_n = n!$

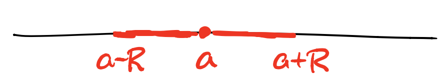
Fact For a power series $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$, 3 possibilities:

1) Series converges only for $x=a$. $\{a\}$



2) Series converges for every value of x. $(-\infty,\infty)$

3) There is some number R such that
the series converges for $|x-a| < R$
the series diverges for $|x-a| > R$



thm, 4 possibilities for the interval of convergence:

$(a-R,\ a+R)$ $\quad$ $[a-R,\ a+R)$

$(a-R,\ a+R]$ $\quad$ $[a-R,\ a+R]$

Q Find the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(2x-3)^n}{n}$.

Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x-3|^{n+1}/n+1}{|2x-3|^n/n} = \frac{|2x-3|^{n+1}}{|2x-3|^n}\cdot\frac{n}{n+1} = |2x-3|\frac{n}{n+1}$$

$$\lim_{n\to\infty} |2x-3|\frac{n}{n+1} = |2x-3|$$

Σ converges if $|2x-3| < 1$

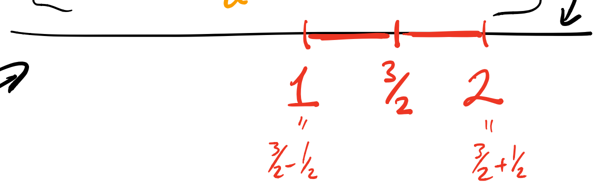
diverges if $|2x-3| > 1$

$\rightarrow$ interval of convergence centered at $2x-3=0$ i.e. $x = 3/2$

$$-1 < 2x-3 < 1$$
$$2 < 2x < 4$$
$$1 < x < 2$$

divide by 2:

$$\Big|x - \underbrace{\tfrac{3}{2}}_{a}\Big| < \underbrace{\tfrac{1}{2}}_{R}$$



So radius of conv. is $R = \frac{1}{2}$

still need to know what happens if $x=1$ or $x=2$.

if $x=1$: $\displaystyle\sum \frac{(2x-3)^n}{n} = \sum \frac{(-1)^n}{n}$ converges by Alt. Series Test

if $x=2$: $\displaystyle\sum \frac{(2x-3)^n}{n} = \sum \frac{1}{n}$ diverges by p-test

So, interval of conv. is $\underline{\underline{[1, 2)}}$

Q $\displaystyle\sum_{n=1}^{\infty} n!\, x^n$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ what's interval of convergence?

$R=0$

0

$\{0\}$

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n}$

$= (n+1)|x|$

$\lim_{n\to\infty} (n+1)|x| = \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

So, ratio test $\Rightarrow \sum$ converges <u>only</u> if $x=0$.

$R=\infty$

$(-\infty, \infty)$

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+2} / 2^{2n+2}((n+1)!)^2}{|x|^{2n} / 2^{2n}(n!)^2}$

$|x|^{2(n+1)}$

$= |x|^2 \cdot \frac{1}{2^2} \cdot \left(\frac{n!}{(n+1)!}\right)^2$

$= \frac{|x|^2}{4} \cdot \frac{1}{(n+1)^2}$

$\lim_{n\to\infty} \frac{|x|^2}{4} \cdot \frac{1}{(n+1)^2} = 0 < 1$

so, $\sum$ converges for <u>all</u> x.

[Stirling: $n! \sim \left(\frac{n}{e}\right)^n$ — could use this to do Root Test on the above example!]

$$\sqrt[n]{\frac{x^{2n}}{2^{2n}} \cdot \frac{1}{(n!)^2}}$$

$$\sim \sqrt[n]{\frac{x^{2n}}{2^{2n}} \cdot \left(\frac{e}{n}\right)^{2n}}$$

$$= \frac{|x|^2}{4} \cdot \left(\frac{e}{n}\right)^2 \longrightarrow 0$$