Exam 3 next Tue Dec 5 7–9pm Jester A121A

covers: multiple integration, sequences/series up to but not including power series

HW 8 – HW 12

(in HW 8 only the multiple-integration part)

HW 13 covered on final exam but not on Exam 3

Functions As Power Series

Using the formula for Σ of a geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

geom. series $a = 1$, $r = x$

(convergent for $|x| < 1$ ie for $x \in (-1,1)$)

ie $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ if $|x| < 1$

Ex What is $\dfrac{1}{0.98}$ to 2 decimal places?

take $x = 0.02$ — then

$$\frac{1}{0.98} = \frac{1}{1-x} = 1 + x + x^2 + \cdots = 1 + (0.02) + (0.02)^2 + \cdots = \underline{1.02}0408\ldots$$

Q A) Find a power series centered at 0 representing $\dfrac{1}{1+x^2}$, and find its interval of convergence.

B) find series for $\dfrac{1}{1+x}$ centered at $\frac{1}{2}$.

A) $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ take $y = -x^2$

then $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$

To find interval of convergence:

<u>one way</u>: Ratio Test — $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}}\right| = |x^2| = x^2$

$\lim_{n\to\infty} x^2 = x^2$

$\longrightarrow$ converges for $x^2 < 1$ i.e. $|x| < 1$

diverges for $x^2 > 1$ i.e. $|x| > 1$

at $x = 1$: $\sum_{n=0}^{\infty} (-1)^n$ diverges by TFD

at $x = -1$: $\sum_{n=0}^{\infty} (-1)^n$ " " "

so interval is $(-1, 1)$

<u>another way</u>: it's geometric with $r = -x^2$

converges just if $|r| < 1$ i.e. $|x| < 1$

$(-1, 1)$

B) $\frac{1}{1+x} = \frac{1}{1 + (x - \frac{1}{2}) + \frac{1}{2}} = \frac{1}{\frac{3}{2} + (x - \frac{1}{2})}$

$= \frac{1}{\frac{3}{2} \cdot \left(1 + \frac{2}{3}(x - \frac{1}{2})\right)}$

$\left[\frac{1}{1+y} = \sum (-y)^n \quad y = \frac{2}{3}(x - \frac{1}{2})\right]$ $= \frac{2}{3} \cdot \frac{1}{1 + \frac{2}{3}(x - \frac{1}{2})} = \frac{2}{3} \cdot \sum_{n=0}^{\infty} \left(-\frac{2}{3}(x - \frac{1}{2})\right)^n$

interval of conv: geometric, $r = -\frac{2}{3}(x-\frac{1}{2})$

so converges just if $|r|<1$ i.e. $\left|-\frac{2}{3}(x-\frac{1}{2})\right|<1$

$$|x-\tfrac{1}{2}|<\tfrac{3}{2}$$

$$-\tfrac{3}{2}<x-\tfrac{1}{2}<\tfrac{3}{2}$$

$$-1<x<2$$

$$(-1,2)$$

Q A) Write $\frac{1}{x+7}$ as power series centered at 0.

$$\frac{1}{1-y}=\sum y^n$$

B) Write $\frac{1}{x+7}$ as power series centered at -2.

A) $$\frac{1}{x+7}=\frac{1}{7}\cdot\frac{1}{\frac{x}{7}+1}=\frac{1}{7}\cdot\frac{1}{1-(-\frac{x}{7})}$$ $\quad y=-\frac{x}{7}$

$$=\frac{1}{7}\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n \leftarrow (-1)^n\cdot x^n\cdot\frac{1}{7^n}$$

$$\left(\text{or: } =\sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}x^n\right)$$

interval of convergence: geometric, $r=-\frac{x}{7}$ so conv. just if $\left|\frac{x}{7}\right|<1$

$$|x|<7$$

$$(-7,7)$$

B) $$\frac{1}{x+7}=\frac{1}{(x+2)+5}=\frac{1}{5}\cdot\frac{1}{\frac{x+2}{5}+1}=\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x+2}{5}\right)^n$$

$$\left(\text{or: } =\sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x+2)^n\right)$$

Q Write $\frac{x^4}{x+7}$ as power series centered at 0.

$$\frac{x^4}{x+7} = x^4 \cdot \frac{1}{x+7} = x^4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^{n+4}$$

[interval of conv: again $(-7,7)$]

or: set $m = n+4$, $n = m-4$

$$\sum_{m=4}^{\infty} \frac{(-1)^{m-4}}{7^{m-3}} x^m$$

$$= \sum_{m=4}^{\infty} \frac{(-1)^m}{7^{m-3}} x^m$$

Q 1) Find a power series for $\frac{1}{(1-x)^2}$ centered at $x=0$.

(Trick: remember $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$)

2) estimate $\frac{1}{(0.98)^2}$ to 2 decimal places.

1) $$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$\frac{d}{dx}(A+B) = \frac{d}{dx}A + \frac{d}{dx}B$$

$$= \sum_{n=0}^{\infty} n x^{n-1}$$

$$= \sum_{n=1}^{\infty} n x^{n-1} = \sum_{m=0}^{\infty} (m+1) x^m$$

$m = n-1$
$n = m+1$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

2) $\dfrac{1}{(0.98)^2} = 1 + 2(0.02) + \cdots \approx \underline{1.04}$

$x = 0.02$

Q 1) Express $\ln(2-x)$ as power series centered at 0. (hint: $\int \frac{-1}{2-x}\,dx = \ln(2-x) + C$)

2) — $\dfrac{1}{(x+7)^2}$ — 0.

1) $\ln(2-x) + C = \displaystyle\int \frac{-1}{2-x}\,dx$

$$-\frac{1}{2-x} = -\frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

$$= \int -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n dx$$

$$= \int -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}x^n\,dx$$

so $\ln(2-x) + C = -\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{x^{n+1}}{n+1}$

$$\ln(2-x) = -\frac{1}{2}\sum \frac{1}{2^n}\frac{x^{n+1}}{n+1} - C$$

to get C, plug in $x=0$: $\ln(2) = -C$

$C = -\ln(2)$

$$\ln(2-x) = \left(-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{x^{n+1}}{n+1}\right) + \ln 2$$

$$= \ln 2 - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}(n+1)}x^{n+1}$$