Rk if $\sum_{n=0}^{\infty} s^{3n+1} = s\sum_{n=0}^{\infty} s^{3n} = s\sum_{n=0}^{\infty} (s^3)^n = s\cdot\frac{1}{1-s^3} = \frac{s}{1-s^3}$

then $\int \sum_{n=0}^{\infty} s^{3n+1}\,ds = \int \frac{s}{1-s^3}\,ds$

$$= \sum_{n=0}^{\infty} \int s^{3n+1}\,ds$$

$$= \sum_{n=0}^{\infty} \frac{s^{3n+2}}{3n+2} + C$$

Last time: functions as power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots \qquad (*)$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = 1-x^2+x^4-x^6+\cdots \qquad (**)$$ by taking $x \to -x^2$ in $(*)$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = 1+2x+3x^2+\cdots$$ by taking $\frac{d}{dx}$ of both sides in $(*)$

$$\ln(1-x) = -\sum \frac{x^{n+1}}{n+1} = -x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$$ by taking $-\int dx$ of both sides of $(*)$

Q 1) Integrate both sides of $(**)$, to get a power series formula for $\tan^{-1}(x)$.

2) Use this to write a formula for π.

1) $\int \frac{1}{1+x^2}dx = \int \sum_{n=0}^{\infty} (-x^2)^n\,dx = \int 1-x^2+x^4-x^6+\cdots\,dx$

$$\tan^{-1}x + C = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

to determine C: plug in $x=0$, then $\tan^{-1}0 + C = 0$

$$0 + C = 0$$

$$C = 0$$

So, $\underline{\underline{\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}}$

radius of convergence $= 1$

2) Plug in $x=1$.

$$\tan^{-1}(1) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1}$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

conditionally conv!

$$\text{So } \pi = 4\cdot \sum_{n=0}^{\infty} (-1)^n/_{2n+1} = 4\cdot\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\cdots\right)$$

Q Find the power series representing $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

centered at $x=0$.

(Hint: already know $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$)

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= \left(-\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}\right) - \left(-\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\right)$$

$$= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right) + \left(\sum_{n=0}^{\infty} (1) \frac{x^{n+1}}{n+1}\right)$$

$$= \sum_{n=0}^{\infty} \left((-1)^n + 1\right) \frac{x^{n+1}}{n+1} = \underset{n=0}{2x} + \underset{n=1}{0} + \underset{n=2}{2\frac{x^3}{3}} + 0 + 2\frac{x^5}{5} + \dots$$

$$= 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

<u>Another way</u>: just take $\frac{d}{dx} \ln\left(\frac{1+x}{1-x}\right) = \frac{1}{1+x} - \frac{1}{1-x}$

$$= \frac{2}{1-x^2}$$

So $\ln\left(\frac{1+x}{1-x}\right) = \int \frac{2}{1-x^2}\,dx$

$$= \int 2\sum_{n=0}^{\infty} (x^2)^n \,dx$$

$$= 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

<u>Exam 3</u>: draft version — 4 problems on multiple $\int$

12 problems on series + sequences

(<u>not</u> power series)

Partial sums: given a sequence

$$\{a_n\} = a_1, a_2, a_3, \ldots$$

the partial sums are

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$

Ex If $a_n = 3$ $\quad \{a_n\} = 3, 3, 3, 3, \ldots$

what is s_n? $\quad s_n = 3n$

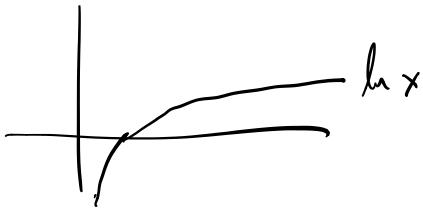
Does $\sum a_n$ converge? No.

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$$

$$\ldots \ll \ln n \ll \cdots \ll \sqrt{n} \ll \cdots n \ll n^2 \ll n^3 \ll \cdots \ll e^n \cdots$$

Q 1) Does $\sum \frac{\ln n}{n^3}$ converge? Yes, it's $< \frac{n}{n^3} = \frac{1}{n^2}$

2) Does $\sum \frac{\ln n}{n}$ converge? No, it's $> \frac{1}{n}$



Q Does $\sum 2\left(\frac{6n+4}{8n+3}\right)^n$ converge?

Yes Root Test: $\sqrt[n]{|a_n|} = \sqrt[n]{2\cdot\left(\frac{6n+4}{8n+3}\right)^n}$

$$= \sqrt[n]{2}\cdot\left(\frac{6n+4}{8n+3}\right)$$

$\lim_{n\to\infty}$: $\sqrt[n]{2} \to 1$, $\left(\frac{6n+4}{8n+3}\right) \to \frac{6}{8}$

$$= \frac{6}{8} < 1 \implies \text{converges}$$

List of tests:

- TFD
- Integral Test
- Root Test
- Comparison Test
- Ratio Test
- Limit Comp Test
- Alt Series Test