Administration

Exam 3 today 7-9pm Jester A121A — up to, not incl power series

Final exam Dec 16 (Sat) 7-10pm — does include power series + Taylor series

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Taylor series

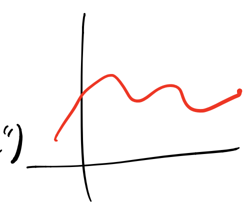
Last few lectures we've been talking about representing functions by series

ex. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\ln(1+2x^2) = -\sum_{n=1}^{\infty} \frac{(-2x^2)^n}{n}$$

How do we get series for other functions?



If we have any function $f(x)$ which is "nice enough" (can be differentiated as many times as we want + "analytic") we can make the Taylor series of $f(x)$ centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

n^{th} derivative of f

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots$$

This series is the series representing $f(x)$!

More exactly: for f nice enough

if this series has radius of convergence $R > 0$

then its sum is $f(x)$ for $x \in (a-R, a+R)$

i.e. $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Q Find the Taylor series for $f(x) = e^x$ around $a=0$ and its radius of convergence.

$f(x) = e^x \to f(0) = 1$

$f'(x) = e^x \to f'(0) = 1$

$f''(x) = e^x \to f''(0) = 1$

$\vdots$

$f^{(n)}(0) = 1$

Taylor series: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Radius of convergence:

ratio test — $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right|$

$\displaystyle = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$

so this $\sum$ always converges

so radius of conv is $R = \infty$

So, $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

e.g. plug in $x=1$: $$e = \sum \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

Q Estimate $e^{\frac{1}{100}}$ to 2 decimal places.

A $$e^{\frac{1}{100}} = 1 + \frac{1}{100} + \frac{\left(\frac{1}{100}\right)^2}{2} + \cdots$$

$$\approx 1.01$$

Q Find the Taylor series for $f(x) = \sin x$ around $x=0$, and its radius of conv.

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$
$\vdots$	$\vdots$

$\rightarrow$ Taylor series: $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

$$\underset{\substack{\uparrow \\ n=0}}{0} + \underset{\substack{\uparrow \\ n=1}}{\frac{1}{1!}x} + 0x^2 + \frac{(-1)}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 + \cdots$$

$\cdots$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

And radius of convergence: $R = \infty$

So: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

Exam review:

Does $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n}$ converge? <u>YES</u>

TFD: $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{n} < \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

<u>Limit-Comparison</u>: $a_n = \frac{\sin(\frac{1}{n})}{n}$ $\quad b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})/n}{1/n^2} = \lim_{n\to\infty} n \sin(\tfrac{1}{n}) = \underline{1} ✓$$

$$\sum \frac{\sin(\frac{1}{n})}{n} \underset{\substack{\uparrow \\ \text{LIM-comp}}}{\approx} \sum \frac{(\frac{1}{n})}{n} = \sum \frac{1}{n^2}$$ which converges by p-test

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Does $\sum_{n=1}^{\infty} \frac{\cos(\frac{1}{n})}{n}$ converge? <u>NO</u>

<u>No</u>: $\sum \frac{\cos(\frac{1}{n})}{n} \underset{\substack{\uparrow \\ \text{LIM-COMP}}}{\approx} \sum \frac{1}{n}$ which diverges by p-test

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$$\sum \frac{\sqrt{n}}{1+n^2} \underset{\substack{\uparrow \\ \text{LIM-comp}}}{\approx} \sum \frac{1}{n^{3/2}}$$ which <u>conv</u> by p-test

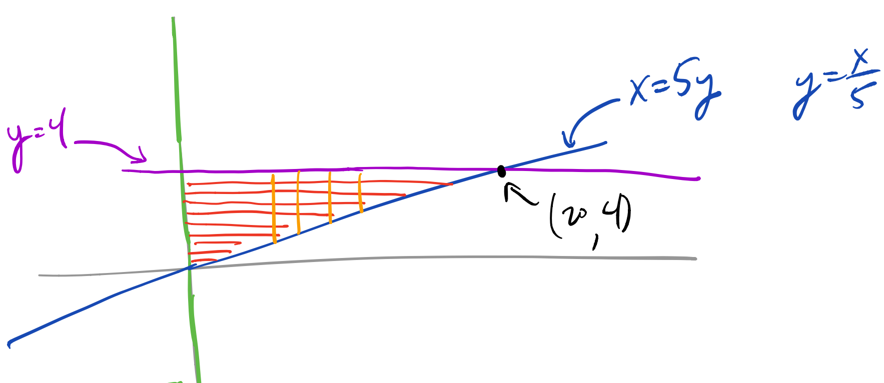
$$\sum (-1)^n \cdot \frac{1+n}{n^2}$$ converges by AST (conditionally)

$$\sum \frac{1}{n \ln n}$$: use integral test ($\longrightarrow$ diverges)

$$\sum \frac{2+\sqrt{n}}{1+\sqrt{n}}$$ diverges by TFD

Reverse the order of integration in

$$\int_{y=0}^{y=4} \int_{x=0}^{x=5y} x^2 y \, dx \, dy$$



$$\int_{x=0}^{x=20} \int_{y=\frac{x}{5}}^{y=4} x^2 y \; dy$$

Find volume of region bounded by $x=0$ $y=0$ $z=0$
$x+y+z=1$

Idea: write this region as $z \leq f(x,y)$

Here: $x+y+z \leq 1 \to z \leq 1-x-y$

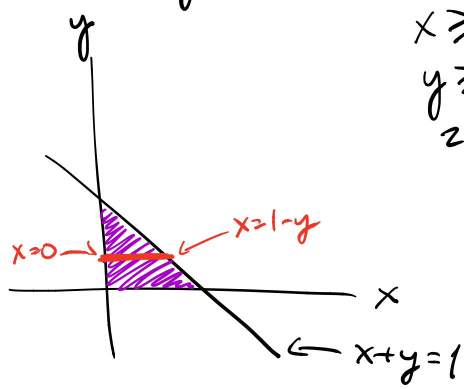
$x \geq 0$

$y \geq 0$

$z \geq 0$

$\to 0 \leq 1-x-y$

$x+y \leq 1$



$$\int_{y=0}^{y=1}\int_{x=0}^{x=1-y} 1-x-y \; dx\, dy$$

$$= \int_{y=0}^{y=1} \left(x - \frac{x^2}{2} - yx \Big|_{x=0}^{x=1-y} \right) dy$$

$$= \int_{y=0}^{y=1} (1-y) - \frac{(1-y)^2}{2} - y(1-y) \; dy$$

$$= \int_{y=0}^{y=1} 1-y - \frac{1-2y+y^2}{2} - y + y^2 \; dy$$

$$= \int_{y=0}^{y=1} \frac{1}{2} - y + \frac{1}{2}y^2 \; dy$$

$\vdots$

Partial sums:

if $S_n = \frac{n^2-2}{3n^2+2}$

then what's ~~the~~ $\sum_{n=1}^{\infty} a_n$?

$$\lim_{n\to\infty} \frac{n^2-2}{3n^2+2} = \frac{1}{3}$$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$$\boxed{\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n}$$

What is $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

geometric: $a = 2$
$r = 1/2$

sum: $\frac{2}{1-\frac{1}{2}} = 4$

$$\frac{2}{1-\frac{1}{2}} = \frac{a}{1-r}$$

sum of n terms of geom series: $S_n = a \cdot \frac{1-r^n}{1-r}$

$$\ln n \ll n^{1/3} \ll n^3 \ll e^n \quad n!$$

$n = 10^{100}$: $n^{1/3} = 10^{30}$

$$\lim_{n \to \infty} \frac{n^{1/3}}{\ln n} \quad \text{diverges}$$

$$\ln n = 100 \ln 10 \approx 300$$

$$\sum \frac{n^{1/3}}{\ln n} \quad \text{diverges} \qquad \sum (-1)^n \frac{n^{1/3}}{\ln n} \quad \text{diverges}$$