Administration

Exam 3 tonight 7–9pm Jester A121A

Final exam Dec 16 7–10pm

covers the whole course including power series, (HW13) Taylor series (practice probs avail. on Quercus)

Taylor series

Last few lectures were about power series representing functions

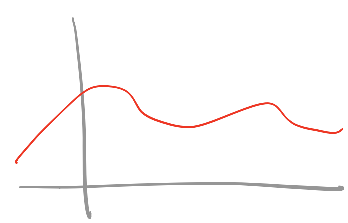
e.g.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$$

Now suppose given a more general function $f(x)$.
How do we find a power series representing $f(x)$?

If we have any $f(x)$ which is smooth



i.e. $f(x)$ is differentiable

$f'(x)$ " "

$f''(x)$ " "

$\vdots$

n-th derivative $f^{(n)}(x)$

then we can make its Taylor series centered at a for any a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If this series has a radius of convergence $R>0$
and $f(x)$ is nice enough ("real analytic")
then for x in $(a-R, a+R)$
we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Q Find the Taylor series for $f(x)=e^x$ around $x=0$.
What is its radius of convergence?

$f(x)=e^x \quad f(0)=1$

$f'(x)=e^x \quad f'(0)=1$

$f''(x)=e^x \quad f''(0)=1$

$\vdots \qquad \vdots$

$f^{(n)}(x)=e^x \quad f^{(n)}(0)=1$

so Taylor series is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n$

e^x is nice enough function $\longrightarrow$ $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots$

Q Estimate $e^{0.01}$ to 2 decimal places.

$$e^{0.01} = 1+0.01+\frac{(0.01)^2}{2}+\frac{(0.01)^3}{6}+\dots$$
$$\approx 1.01$$

To find radius of conv: <u>Ratio Test</u> $\sum \frac{x^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \frac{|x|}{n+1}$$

$$\lim_{n\to\infty} \frac{|x|}{n+1} = 0 \qquad 0<1 \longrightarrow \text{sum converges}$$

(no matter what x is)

So $R=\infty$

so $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

$$\boxed{\sum \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Ex Find the Taylor series for $f(x) = \sin x$ around $x=0$

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$
$\vdots$	

$0, 1, 0, -1, 0, 1, 0, -1, \ldots$

$n=0 \quad n=1 \quad n=3 \quad n=5$

So, Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \frac{1}{1!}x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{-1}{7!}x^7 + \cdots$

$$= \underline{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}} \left(= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} \right)$$

Volume bounded by: $x=0$, $y=0$, $z=0$, $x+y+z=3$?

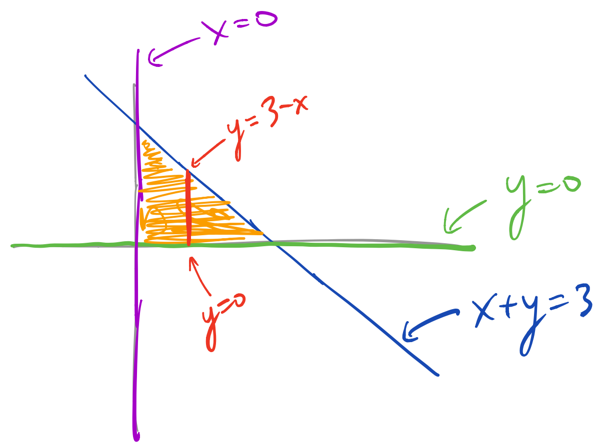
$x \geq 0$, $y \geq 0$, $z \geq 0$, $x+y+z \leq 3$ $\qquad z \leq 3-x-y$

think of this as region <u>under</u> $z=f(x,y)$ $\qquad f(x,y)=3-x-y$

<u>over</u> $z=0$

the region of integration in x-y plane: $3-x-y \geq 0$

$x+y \leq 3$

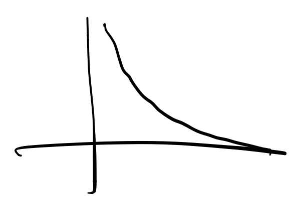


$$\iint 3-x-y \, dA$$

$$= \int_{x=0}^{x=3} \left[\int_{y=0}^{y=3-x} 3-x-y \, dy \right] dx$$

$\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{3+2\sqrt{n}}$ <u>diverges</u> by TFD

$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ <u>diverges</u> by $\int$ test



$\sum (-1)^n \frac{1+\sqrt{n}}{n}$ <u>conditional</u> conv.

(<u>not</u> abs. convergent b/c $> \frac{1}{n}$)

(is convergent, by alt. series test)

$$\lim_{n\to\infty} \frac{n}{\ln n} = \underline{\text{divergent}} \quad (\text{by L'Hospital})$$

$$1 \ll \ln\ln n \ll \ln n \ll n^{1/2} \ll n \ll n^2 \ll e^n \ll n! < n^n$$

$$\sum_{k=1}^{\infty} \frac{1+3^k}{4^k} = \sum_{k=1}^{\infty} \frac{1}{4^k} + \sum_{k=1}^{\infty} \frac{3^k}{4^k}$$

geom: $a = \frac{1}{4}$, $r = \frac{1}{4}$ $\qquad$ $a = \frac{3}{4}$, $r = \frac{3}{4}$

$$\longrightarrow \frac{\frac{1}{4}}{1-\frac{1}{4}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}$$

$$= \frac{1}{3} + 3 = \underline{\underline{\frac{10}{3}}}$$

$$a_n = \left(\frac{n+2}{n-4}\right)^n \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^n}{\left(1-\frac{4}{n}\right)^n}$$

$$= \frac{e^2}{e^{-4}} = \underline{\underline{e^6}}$$

$$a_n = \left(1-\frac{3}{4n}\right)^{2n} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[\left(1-\frac{(3/4)}{n}\right)^n\right]^2$$

$$= \left(e^{-3/4}\right)^2$$

$$= \underline{\underline{e^{-3/2}}}$$

$\sum_{n=1}^{\infty} \frac{n^n}{n!}$: $\underline{\text{ratio test}}$

$$\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \frac{n+1 \cdot \frac{(n+1)^n}{n^n}}{n+1} \to \underline{\underline{e}}$$

$\to$ diverges

$$\sum \frac{n^2+6n}{3n^4+8n} \approx \sum \frac{1}{3n^2} \Rightarrow \text{converges}$$

Lim-Comp

$$\sum \frac{1}{n^3 \ln n} < \sum \frac{1}{n^3} \Rightarrow \text{converges}$$

Comp.

$$\cos(\pi n) = (-1)^n$$

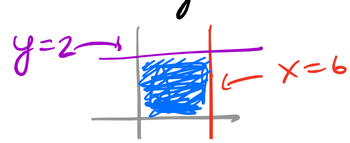
$$\sum \cos(\pi n) \cdot \frac{(\ln n)^2}{n}$$

converges conditionally

$x = 6$

$z = 4 - y^2$ $\quad z \geq 0 \quad 4 - y^2 \geq 0$

$x \geq 0$
$y \geq 0$



$$\int_{y=0}^{y=2}\int_{x=0}^{x=6} 4-y^2\,dx\,dy$$

$$a_n = \cos(\pi n)\cdot \ln\left(\frac{An-1}{1+3n}\right)$$

$$= (-1)^n \cdot \boxed{\ln\left(\frac{An-1}{3n+1}\right)} \to \ln\left(\frac{A}{3}\right)$$

diverges by TFD if $A \neq 3$

converges by AST if $A = 3$

$$\int_{y=0}^{y=1}\int_{x=1}^{x=2} \frac{3x^2+y}{y+1} + \frac{y}{x}\,dx\,dy$$

$$\int_{y=0}^{y=1}\left(\int_{x=1}^{x=2}\frac{y}{x}\,dx\right)dy = \int_{y=0}^{y=1}\left(y\ln x\Big|_{x=1}^{x=2}\right)dy$$

$$= \int_{y=0}^{y=1} y\ln 2\;dy = \frac{1}{2}y^2\Big|_{y=0}^{y=1}$$

$$= \frac{1}{2}\ln 2$$

$= \ldots$

$$\int_2^7 f(x)\,dx < \sum_{n=2}^{6} f(n) = f(2)+f(3)+\cdots+f(6)$$