Lecture 10

30 Sep 2014

Reminder: I will have office hour today 1:00–2:00pm

I will have no office hour tomorrow

Don will have office hour tomorrow (Wed) 3:00–4:30pm

Exam 1 Thursday

You only need pencils + erasers

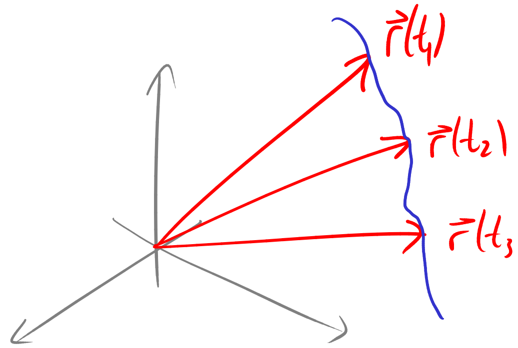
(Calculators are not permitted)

Grades should be posted Monday evening

15 questions

Vector functions (Ch 13.1 cont'd)

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$



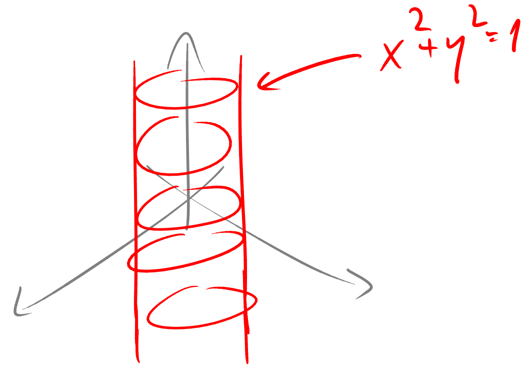
Recall: vector functions are closely related to parameterized curves — if put the tails of all vectors $\vec{r}(t)$ at the origin, the tips of $\vec{r}(t)$ sweep out a curve.

From now on, sloppily say vector f^n = parameterized curve.

Ex Write a parameterization for the intersection between (in 3 dim) the locus $x^2+y^2=1$ cylinder

and $y+z=2$ plane

Expect this intersection will look like an ellipse.



Idea: $z = 2 - y$ so in particular, z is totally determined by y.

So, let's deal with x and y first, then come back to z.

To parametrize x and y, say $\quad x = \cos t$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad y = \sin t$

then using $z = 2 - y$ have $z = 2 - \sin t$

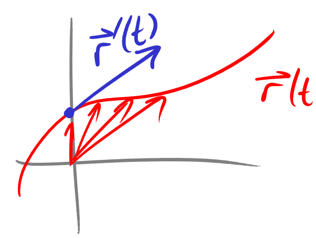
So altogether $\vec{r} = \langle x(t), y(t), z(t) \rangle = \underline{\underline{\langle \cos(t), \sin(t), 2 - \sin(t) \rangle}}$

Calculus of Vector Functions, cont'd (Ch 13.2)

Last time: <u>differentiation</u> of vector functions

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \qquad\qquad \vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

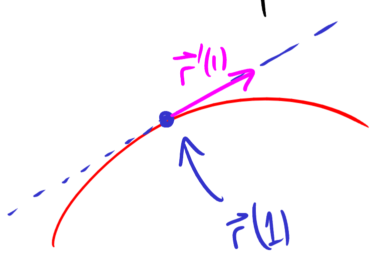
<u>Interpretation</u>: $\vec{r}\,'(t)$ is <u>tangent vector</u> to the curve $\vec{r}(t)$



<u>Ex</u> Find the tangent line to the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$.

(in parametric form)

Need a point on the tangent line and a vector pointing along the tangent line.



Point on the line: $\vec{r}(1) = \langle 1, 1, 1 \rangle$

Vector along the line: $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

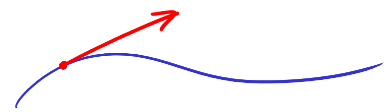
so $\vec{r}\,'(1) = \langle 1, 2, 3 \rangle$

So the tangent line is $\vec{l}(t) = \vec{r}(1) + t\,\vec{r}\,'(1)$

$$= \langle 1,1,1\rangle + t\langle 1,2,3\rangle$$
$$= \langle 1+t,\ 1+2t,\ 1+3t\rangle$$

Velocity

If we have a path $P(t)$ with displacement from the origin given by a vector $\vec{r}(t)$ then $\vec{r}\,'(t) = \vec{v}(t)$ is the velocity



The length $\|\vec{v}(t)\|$ is the speed

(These definitions apply either in 2-d or in 3-d)

Ex If $\vec{r}(t) = \langle t, t^2+1\rangle$

then $\vec{r}\,'(t) = \langle 1, 2t\rangle$

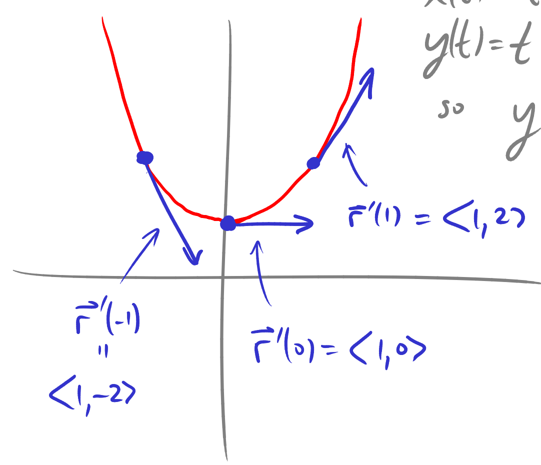
e.g. if $t=0$, $\vec{r}\,'(0) = \langle 1,0\rangle$

$\vec{r}(0) = \langle 0,1\rangle$

$t=1$, $\vec{r}\,'(1) = \langle 1,2\rangle$

$t=-1$, $\vec{r}\,'(-1) = \langle 1,-2\rangle$

$x(t)=t$
$y(t)=t^2+1$
so $y=x^2+1$



Speed: $\|\vec{r}\,'(t)\| = \|\langle 1,2t\rangle\|$
$= \sqrt{1+4t^2}$

Differentiation rules

$$\frac{d}{dt}\left(\vec{u}(t)+\vec{v}(t)\right) = \vec{u}\,'(t)+\vec{v}\,'(t)$$

$$\frac{d}{dt}\left(c\cdot\vec{u}(t)\right) = c\cdot\frac{d}{dt}\vec{u}(t)$$

$$\frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$$

$$\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$$

careful with order! $\longrightarrow$ $$\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$$

$$\frac{d}{dt}\left(\vec{u}(f(t))\right) = f'(t)\,\vec{u}'(f(t))$$ [Chain Rule for vector functions]

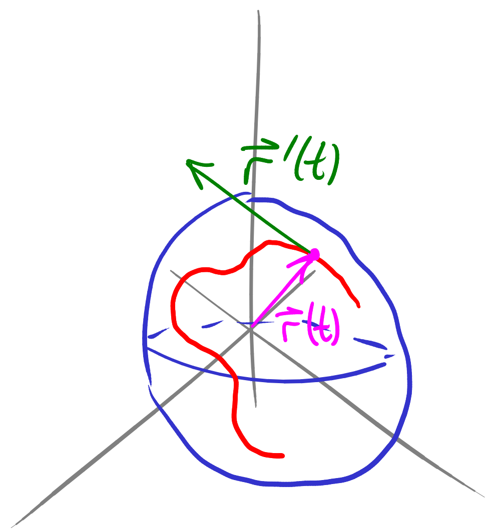
[Proofs of these in text: by reducing to the rules for ordinary functions]

Ex If $\vec{u}(t) = \vec{a} + t\vec{b} + t^2\vec{c}$

then $\vec{u}'(t) = \vec{b} + 2t\vec{c}$ "just as if $\vec{a}, \vec{b}, \vec{c}$ were constant #'s"

Ex Prove: if $\vec{r}(t)$ lies (for all t) on a sphere of radius c around the origin,

then $\vec{r}(t) \perp \vec{r}'(t)$.



We know that

$$\sqrt{x(t)^2 + y(t)^2 + z(t)^2} = c$$

distance from (x,y,z) to $(0,0,0)$

i.e. $$\|\vec{r}(t)\| = c$$

$$\|\vec{r}(t)\|^2 = c^2$$

$$\vec{r}(t)\cdot\vec{r}(t) = c^2$$

take $\frac{d}{dt}$ of both sides: $$\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \frac{d}{dt}(c^2)$$

$$\vec{r}'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}'(t) = 0$$

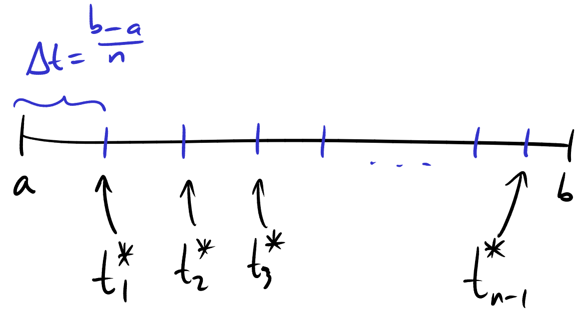
$$2\,\vec{r}'(t)\cdot\vec{r}(t) = 0$$

$$\vec{r}'(t)\cdot\vec{r}(t) = 0$$ ie $\underline{\vec{r}'(t) \perp \vec{r}(t).}$

Integrals of vector functions

We define the integral of a vector function much like we did for scalar functions:

by Riemann sums (not as area under something!)



$$\int_a^b \vec{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{r}(t_i^*)\,\Delta t$$

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then $\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$

Key fact: Fund^l Thm of Calculus for vector functions

If $\vec{R}'(t) = \vec{r}(t)$ then $\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$.

Ex If $\vec{r}(t) = \langle 1, e^t, t^2 \rangle$

then what is $\int_0^2 \vec{r}(t)\,dt$?

$\vec{r}(t) = \vec{R}'(t)$ where

constant vector

$$\vec{R}(t) = \langle t, e^t, \tfrac{1}{3}t^3 \rangle + \vec{C}$$

$$\int_0^2 \vec{r}(t)\,dt = \vec{R}(2) - \vec{R}(0) = \left(\langle 2, e^2, \tfrac{8}{3} \rangle + \vec{C}\right) - \left(\langle 0, 1, 0 \rangle + \vec{C}\right)$$

$$\left(= \vec{R}(t)\Big]_0^2\right) \qquad = \underline{\underline{\langle 2, e^2 - 1, \tfrac{8}{3} \rangle}}$$

(We'd get the same answer by $\int_0^2$ each component separately)

Indefinite integrals of vector functions:

given $\vec{r}(t)$, $\int \vec{r}(t)\,dt$ means any antiderivative of $\vec{r}(t)$.

Ex $\int (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}\,dt$

$$= (\sin t)\vec{i} + (-\cos t)\vec{j} + \frac{1}{2}t^2\vec{k} + \vec{C}$$

Ex Suppose a particle moves with velocity

$$\vec{v}(t) = \langle t, e^t, te^t \rangle$$

and at $t=0$ the particle is at $(1,1,1)$.

Find the position $\vec{r}(t)$.

$\vec{v}(t) = \vec{r}\,'(t)$ i.e. $\vec{r}(t) = \int \vec{v}(t)\,dt$

$$= \int \langle t, e^t, te^t \rangle\,dt$$

(int. by parts)

$$= \langle \tfrac{1}{2}t^2, e^t, (t-1)e^t \rangle + \vec{C}$$

$\vec{r}(0) = \langle 0, 1, -1 \rangle + \vec{C}$ and we know $\vec{r}(0) = \langle 1,1,1 \rangle$

so $\vec{C} + \langle 0,1,-1 \rangle = \langle 1,1,1 \rangle$

$$\vec{C} = \langle 1,1,1 \rangle - \langle 0,1,-1 \rangle = \langle 1,0,2 \rangle$$

So $\vec{r}(t) = \langle \frac{1}{2}t^2, e^t, (t-1)e^t \rangle + \langle 1, 0, 2 \rangle$

$$= \langle \tfrac{1}{2}t^2 + 1, e^t, (t-1)e^t + 2 \rangle$$